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Calabi-Yau metrics for quotients and complete intersections

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Abstract: We extend previous computations of Calabi-Yau metrics on projective hypersurfaces to free quotients, complete intersections, and free quotients of complete intersections. In particular, we construct these metrics on generic quintics, four-generation quotients of the quintic, Schoen Calabi-Yau complete intersections and the quotient of a Schoen manifold with $\mathbb{Z}_3 \times \mathbb{Z}_3$ fundamental group that was previously used to construct a heterotic standard model. Various numerical investigations into the dependence of Donaldson's algorithm on the integration scheme, as well as on the Kähler and complex structure moduli, are also performed.

Keywords: Space-Time Symmetries, Superstrings and Heterotic Strings, Superstring Vacua, Differential and Algebraic Geometry.

Contents

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1. Introduction

A central problem of string theory is to find compactifications which can reproduce real world physics, in particular the Standard Model. The first and still one of the best motivated ways to achieve this are heterotic string compactifications on Calabi-Yau manifolds [1]. In particular, the so-called "non-standard embedding" of $E_8 \times E_8$ heterotic strings has been a very fruitful approach towards model building.

For a variety of reasons, the most successful models of this type to date are based on non-simply connected Calabi-Yau threefolds. These manifolds admit discrete Wilson lines which, together with a non-flat vector bundle, play an important role in breaking the heterotic E_8 gauge theory down to the Standard Model [2 – 7]. In the process, they project out many unwanted matter components. In particular, one can use this mechanism to solve the doublet-triplet splitting problem [8, 9]. Finally, the non-simply connected threefolds have many fewer moduli as compared to their simply connected covering spaces [10]. In recent work [11 – 14], three generation models with a variety of desirable features were introduced. These are based on a certain quotient of the Schoen Calabi-Yau threefold, which yields a non-simply connected Calabi-Yau manifold with fundamental group $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Ultimately, it would be desirable to compute all of the observable quantities of particle physics, in particular gauge and Yukawa couplings, from the microscopic physics of string theory [15, 16]. There are many issues which must be addressed to do this. Physical Yukawa couplings, for example, depend on both coefficients in the superpotential and the explicit form of the Kähler potential. In a very limited number of specific geometries [17 – 20], the former can be computed using sophisticated methods of algebraic geometry, topological string theory and the like. For the latter, we generally have only the qualitative statement that a coefficient is "expected to be of order one". Doing better and, in particular, extending these calculations to non-standard embedding, multiply connected compactifications has been an outstanding problem [1].

In recent work [21, 22], a plan has been outlined to analyze these problems numerically, at least in the classical limit. The essential point is that, today, there are good enough algorithms and fast enough computers to calculate Ricci-flat metrics and to solve the hermitian Yang-Mills equation for the gauge connection directly. Given this data, one can then find the correctly normalized zero modes of fields, determine the coefficients in the superpotential and compute the explicit form of the Kähler potential. Some progress in this direction was made in [21 – 25], and also [26 – 28].

In the present work, we take some significant steps forward in computing Calabi-Yau metrics. Some of the steps are technical and computational improvements, which we will discuss. But the primary new ingredient is the ability to solve for Ricci-flat metrics on non-simply connected Calabi-Yau manifolds. While in broad conceptual terms the procedure is similar to the simply connected case, in practice the problem of finding and working with a complete basis of holomorphic sections of a line bundle, as used in Donaldson's method, is now quite intricate. To solve this, we systematically use the techniques of Invariant Theory [29].

We begin, in section 2, by extending Donaldson's algorithm to the computation of

Calabi-Yau metrics for generic quintic threefolds. This formalism is first applied to the simple Fermat quintic, reproducing and extending the results of [23]. We then numerically calculate the Calabi-Yau metrics, and test their Ricci-flatness, for a number of random points in the complex structure moduli space. All of these manifolds are, of course, simply connected. We then proceed to non-simply connected manifolds or, equivalently, to covering spaces that admit fixed point free group actions. In section 3, we outline the general idea and review some of the Invariant Theory, in particular the Poincaré series, Molien formula and the Hironaka decomposition, that we will use. This formalism will then be applied in section 4 to the subset of quintics that admit a $\mathbb{Z}_5 \times \mathbb{Z}_5$ fixed point free group action. Using the Molien formula and the Hironaka decomposition, we determine the $\mathbb{Z}_5 \times \mathbb{Z}_5$ invariant sections on such quintics and, hence, on the $\mathbb{Z}_5 \times \mathbb{Z}_5$ multiply connected quotient space. Using these, we can extend Donaldson's algorithm and compute the Calabi-Yau metric, and test its Ricci-flatness, on the quotient. As a by-product of this process, we note that there are now two ways to compute a Calabi-Yau metric on the covering space; first, as a point in the quintic moduli space employing the methods of section 2 and second, by using Donaldson's algorithm for invariant sections only. These two methods are compared in section 4. Note that only the second approach descends to the $\mathbb{Z}_5 \times \mathbb{Z}_5$ quotient threefolds.

In section 5, we describe Schoen threefolds. We show how to compute Calabi-Yau metrics on these simply connected complete intersection manifolds and, as always, test their Ricci-flatness. Schoen manifolds which admit a fixed point free $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action are then discussed in section 6. Proceeding as in section 4, the Molien formula and the Hironaka decomposition are generalized to complete intersections. These are then used to find all $\mathbb{Z}_3 \times \mathbb{Z}_3$ invariant sections. These descend to the $\mathbb{Z}_3 \times \mathbb{Z}_3$ quotient, and are used to compute the Calabi-Yau metric on this multiply connected threefold.

In addition, we explicitly check algebraic independence of primary invariants, which are defined in section 3, for quintics in section A. Finally some portions of the code used in this paper are presented in section B.

2. The quintic

2.1 Parametrizing metrics

Quintics are Calabi-Yau threefolds $\widetilde{Q} \subset \mathbb{P}^4$. As usual, the five homogeneous coordinates $[z_0 : z_1 : z_2 : z_3 : z_4]$ on $\mathbb{P}^4$ are subject to the identification

$$[z_0 : z_1 : z_2 : z_3 : z_4] = [\lambda z_0 : \lambda z_1 : \lambda z_2 : \lambda z_3 : \lambda z_4] \quad \forall \lambda \in \mathbb{C} - \{0\}. \tag{2.1}$$

In general, a hypersurface in $\mathbb{P}^4$ is Calabi-Yau if and only if it is the zero locus of a degree-5 homogeneous polynomial[1]

$$\widetilde{Q}(z) = \sum_{n_0+n_1+n_2+n_3+n_4=5} c_{(n_0,n_1,n_2,n_3,n_4)} z_0^{n_0} z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4}. \tag{2.2}$$

[1]Hence the name quintic.

Note that, abusing notation, we denote both the threefold and its defining polynomial by $\widetilde{Q}$. There are $\binom{5+4-1}{4} = 126$ degree-5 monomials, leading to 126 coefficients $c_{(n_0,n_1,n_2,n_3,n_4)} \in \mathbb{C}$. These can be reduced by redefining the z_i-coordinates under $GL(5,\mathbb{C})$. Hence, the number of complex structure moduli of a generic $\widetilde{Q}$ is $126 - 25 = 101$. A particularly simple point in this moduli space is the so-called Fermat quintic $\widetilde{Q}_F$, defined as the zero-locus of

$$\widetilde{Q}_F(z) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5. \tag{2.3}$$

We will return to the Fermat quintic later in this section.

In general, the metric on a real six-dimensional manifold is a symmetric two-index tensor, having 21 independent components. However, on a Calabi-Yau (more generally, a Kähler) manifold the metric has fewer independent components. First, in complex coordinates, the completely holomorphic and completely anti-holomorphic components vanish,

$$g_{ij}(z,\bar{z}) = 0, \quad g_{\bar{\imath}\bar{\jmath}}(z,\bar{z}) = 0. \tag{2.4}$$

Second, the mixed components are the derivatives of a single function

$$g_{i\bar{\jmath}}(z,\bar{z}) = g^*_{\bar{\imath}j}(z,\bar{z}) = \partial_i \bar{\partial}_{\bar{\jmath}} K(z,\bar{z}). \tag{2.5}$$

The hermitian metric $g_{i\bar{\jmath}}$ suggests the following definition of a real $(1,1)$-form, the Kähler form

$$\omega = \frac{i}{2} g_{i\bar{\jmath}}\, \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_{\bar{\jmath}} = \frac{i}{2} \partial\bar{\partial} K(z,\bar{z}). \tag{2.6}$$

The Kähler potential $K(z,\bar{z})$ is locally a real function, but not globally; on the overlap of coordinate charts one has to patch it together by Kähler transformations

$$K(z,\bar{z}) \sim K(z,\bar{z}) + f(z) + \bar{f}(\bar{z}). \tag{2.7}$$

The metric eq. (2.5) is then globally defined.

The 5 homogeneous coordinates on $\mathbb{P}^4$ clearly come with a natural SU(5) action, so a naive ansatz for the Kähler potential would be invariant under this symmetry. However, the obvious SU(5)-invariant $|z_0|^2 + \cdots + |z_4|^2$ would not transform correctly under the rescaling eq. (2.1) with $\lambda = \lambda(z)$. Therefore, one is led to the unique[2] SU(5) invariant Kähler potential

$$K_{\mathrm{FS}} = \frac{1}{\pi} \ln \sum_{i=0}^{4} z_i \bar{z}_{\bar{\imath}}. \tag{2.9}$$

One can slightly generalize this by inserting an arbitrary hermitian 5×5 matrix $h^{\alpha\bar{\beta}}$,

$$K_{\mathrm{FS}} = \frac{1}{\pi} \ln \sum_{\alpha,\bar{\beta}=0}^{4} h^{\alpha\bar{\beta}} z_\alpha \bar{z}_{\bar{\beta}}. \tag{2.10}$$

[2] Unique up to an overall scale, of course. The scale is fixed by demanding that ω_{FS} is an integral class, $\omega \in H^2(\mathbb{P}^4,\mathbb{Z})$. To verify the integrality, observe that the volume integral over the curve $[1:t:0:0:0]$ in $\mathbb{P}^4$ is

$$\int_{\mathbb{C}} \frac{i}{2} \partial\bar{\partial} K_{\mathrm{FS}}\big([1:t:0:0:0]\big) = \int_{\mathbb{C}} \frac{1}{\pi} \partial_t \bar{\partial}_{\bar{t}} \ln(1+t\bar{t}) \frac{i}{2}\, \mathrm{d}t\, \mathrm{d}\bar{t} = 1. \tag{2.8}$$

Any Kähler potential of this form is called a Fubini-Study Kähler potential (giving rise to a Fubini-Study metric). At this point the introduction of an arbitrary hermitian $h^{\alpha\bar{\beta}}$ does not yield anything really new, as one can always diagonalize it by coordinate changes. However, strictly speaking, different $h^{\alpha\bar{\beta}}$ are different Fubini-Study metrics.

The above Kähler potential is defined on the whole $\mathbb{P}^4$ and, hence, defines a metric on $\mathbb{P}^4$. But this induces a metric on the hypersurface $\widetilde{Q}$, whose Kähler potential is simply the restriction. Unfortunately, the restriction of the Fubini-Study metric to the quintic is far from Ricci-flat. Indeed, not a single Ricci-flat metric on any proper Calabi-Yau threefold is known. One of the reasons is that proper Calabi-Yau metrics have no continuous isometries, so it is inherently difficult to write one down analytically. Recently, Donaldson presented an algorithm for numerically approximating Calabi-Yau metrics to any desired degree [21]. To do this in the quintic context, take a "suitable" generalization, that is, one containing many more free parameters of the Fubini-Study metric derived from eq. (2.10) on $\mathbb{P}^4$. Then restrict this ansatz to $\widetilde{Q}$ and numerically adjust the parameters so as to approach the Calabi-Yau metric. An obvious idea to implement this is to replace the degree-1 monomials z_α in eq. (2.10) by higher degree-k monomials, thus introducing many more coefficients in the process. However, note that the degree k is the Kähler class

$$k \in H^{1,1}(\mathbb{P}^4, \mathbb{Z}) \simeq \mathbb{Z}. \tag{2.11}$$

The reason for this is clear, for example, if we multiply K_{FS} in eq. (2.9) by k. Then

$$k\, K_{\mathrm{FS}} = \frac{k}{\pi} \ln \sum_{i=0}^{4} z_i \bar{z}_i = \frac{1}{\pi} \ln \sum_{i_1,\ldots,i_k=0}^{4} z_{i_1} \cdots z_{i_k} \bar{z}_{\bar{\imath}_1} \cdots \bar{z}_{\bar{\imath}_k}. \tag{2.12}$$

Hence, if we want to keep the overall volume fixed, the correctly normalized generalization of eq. (2.10) is

$$K(z,\bar{z}) = \frac{1}{k\pi} \ln \sum_{\substack{i_1,\ldots,i_k=0 \\ \bar{\jmath}_1,\ldots,\bar{\jmath}_k=0}}^{4} h^{(i_1,\ldots,i_k),(\bar{\jmath}_1,\ldots,\bar{\jmath}_k)} \underbrace{z_{i_1} \cdots z_{i_k}}_{\text{degree } k} \underbrace{\bar{z}_{\bar{\jmath}_1} \cdots \bar{z}_{\bar{\jmath}_k}}_{\text{degree } k}. \tag{2.13}$$

Note that the monomials $z_{i_1} \cdots z_{i_k}$, where $i_1, \ldots, i_k = 0, \ldots, 4$ and integer $k \geq 0$, are basis vectors for the space of polynomials $\mathbb{C}[z_0, \ldots, z_k]$. For fixed total degree k, they span the subspace $\mathbb{C}[z_0, \ldots, z_k]_k$ of dimension

$$\hat{N}_k = \binom{5+k-1}{k}. \tag{2.14}$$

Some values of $\hat{N}_k$ are given in table 1. In particular, the matrix of coefficients h now must be a hermitian $\hat{N}_k \times \hat{N}_k$ matrix.

However, there is one remaining issue as soon as $k \geq 5$, namely, that the monomials will not necessarily remain independent when restricted to $\widetilde{Q}$. In order to correctly parametrize the degrees of freedom on $\widetilde{Q}$, we have to pick a basis for the quotient

$$\mathbb{C}\left[z_0, \ldots, z_4\right]_k \Big/ \left\langle \widetilde{Q}(z) \right\rangle \tag{2.15}$$

k	1	2	3	4	5	6	7	8
$\hat{N}_k$	5	15	35	70	126	210	330	495
N_k	5	15	35	70	125	205	315	460

Table 1: The number of homogeneous polynomials $\hat{N}_k$ and the number of remaining polynomials N_k after imposing the hypersurface constraint, see eq. (2.16).

for the degree-k polynomials modulo the hypersurface equation. Let us denote this basis by s_α, $\alpha = 0, \ldots, N_k - 1$. It can be shown that for any quintic

$$N_k = \begin{cases} \hat{N}_k = \binom{5+k-1}{k} & 0 \leq k < 5 \\ \hat{N}_k - \hat{N}_{k-5} = \binom{5+k-1}{k} - \binom{k-1}{k-5} & k \geq 5. \end{cases} \tag{2.16}$$

Some values of the N_k are listed in table 1. For any given quintic polynomial $\widetilde{Q}(z)$ and degree k, computing an explicit polynomial basis $\{s_\alpha\}$ is straightforward. As an example, let us consider the the Fermat quintic defined by the vanishing of $\widetilde{Q}_F(z)$, see eq. (2.3). In this case, a basis for the quotient eq. (2.15) can be found by eliminating from any polynomial in $\mathbb{C}[z_0, \ldots, z_4]_k$ all occurrences of z_0^5 using $z_0^5 = -(z_1^5 + z_2^5 + z_3^5 + z_4^5)$.

Using the basis s_α for the quotient ring, one finally arrives at the following ansatz

$$K_{h,k} = \frac{1}{k\pi} \ln \sum_{\alpha,\bar{\beta}=0}^{N_k-1} h^{\alpha\bar{\beta}} s_\alpha \bar{s}_{\bar{\beta}} = \frac{1}{k\pi} \ln \|s\|_{h,k}^2 \tag{2.17}$$

for the Kahler potential and, hence, the approximating metric. Note that they are formally defined on $\mathbb{P}^4$ but restrict directly to $\widetilde{Q}$, by construction. Obviously, this is not the only possible ansatz for the approximating metric, and the reason for this particular choice will only become clear later on. However, let us simply mention here that there is a rather simple iteration scheme [23, 24, 30] involving $K_{h,k}$ which will converge to the Ricci-flat metric in the limit $k \to \infty$. Note that, in contrast to the Fubini-Study Kähler potential eq. (2.10), the matrix $h^{\alpha\bar{\beta}}$ in eq. (2.17) cannot be diagonalized by a $GL(5, \mathbb{C})$ coordinate change on the ambient $\mathbb{P}^4$ for $k \geq 2$.

Let us note that there is a geometric interpretation of the homogeneous polynomials, which will be important later on. Due to the rescaling ambiguity eq. (2.1), the homogeneous polynomials are not functions on $\mathbb{P}^4$, but need to be interpreted as sections of a line bundle. The line bundle for degree-k polynomials is denoted $\mathcal{O}_{\mathbb{P}^4}(k)$ and, in particular, the homogeneous coordinates are sections of $\mathcal{O}_{\mathbb{P}^4}(1)$. In general, the following are the same

- Homogeneous polynomials of degree k in n variables.
- Sections of the line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(k)$.

Moreover, the quotient of the homogeneous polynomials by the quintic, eq. (2.15), is geometrically the restriction of the line bundle $\mathcal{O}_{\mathbb{P}^4}(k)$ to the quintic hypersurface. That is, start with the identification above,

$$H^0\big(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(k)\big) = \mathbb{C}[z_0, z_1, z_2, z_3, z_4]_k. \tag{2.18}$$

After restricting the sections of $\mathcal{O}_{\mathbb{P}^4}(k)$ to $\widetilde{Q}$, they satisfy the relation $\widetilde{Q}(z) = 0$. Hence, the restriction is

$$\begin{aligned} H^0\big(\widetilde{Q}, \mathcal{O}_{\widetilde{Q}}(k)\big) &= \mathbb{C}[z_0, z_1, z_2, z_3, z_4]_k \Big/ \Big\langle \widetilde{Q}(z) \Big\rangle_k \\ &= \mathbb{C}[z_0, z_1, z_2, z_3, z_4]_k \Big/ \Big(\widetilde{Q}\, \mathbb{C}[z_0, z_1, z_2, z_3, z_4]_{k-5} \Big). \end{aligned} \tag{2.19}$$

More technically, this whole discussion can be represented by the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0\big(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(k-5)\big) \arrow[r, "\times \widetilde{Q}(z)"] \arrow[d, equal] & H^0\big(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(k)\big) \arrow[r, "\text{restrict}"] \arrow[d, equal] & H^0\big(\widetilde{Q}, \mathcal{O}_{\widetilde{Q}}(k)\big) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{C}\,[z_0, \ldots, z_4]_{k-5} \arrow[r, "\times \widetilde{Q}(z)"] & \mathbb{C}\,[z_0, \ldots, z_4]_k \arrow[r] & \mathbb{C}[z_0, \ldots, z_4]_k / \Big\langle \widetilde{Q}(z) \Big\rangle_k \arrow[r] & 0 \end{tikzcd} \tag{2.20}$$

2.2 Donaldson's algorithm

Once we have specified the form for the Kähler potential, our problem reduces to finding the "right" matrix $h^{\alpha\bar{\beta}}$. This leads us to the notion of T-map and balanced metrics, which we now introduce. First, note that eq. (2.17) provides a way to define an inner product of two sections. While it makes sense to evaluate a function at a point, one cannot "evaluate" a section (a homogeneous polynomial) at a point since the result would only be valid up to an overall scale.[3] However, after picking $\|s\|^2_{h,k}$, see eq. (2.17), one can cancel the scaling ambiguity and define

$$(S, S')(p) = \frac{S(p)\,\bar{S}'(p)}{\|s\|^2_{h,k}(p)} = \frac{S(p)\,\bar{S}'(p)}{\sum_{\alpha,\bar{\beta}} h^{\alpha\bar{\beta}}\, s_\alpha(p)\, \bar{s}_{\bar{\beta}}(p)} \quad \forall p \in \widetilde{Q} \tag{2.21}$$

for arbitrary sections (degree-k homogeneous polynomials) S, $S' \in H^0\big(\widetilde{Q}, \mathcal{O}_{\widetilde{Q}}(k)\big)$. Note that the $s_0, \ldots, s_{N_k-1}$ are a basis for the space of sections, so there are always constants $c^\alpha \in \mathbb{C}$ such that

$$S = \sum_{\alpha=0}^{N_k-1} c^\alpha s_\alpha. \tag{2.22}$$

The point-wise hermitian form $(\ ,\ )$ is called a metric on the line bundle $\mathcal{O}_{\widetilde{Q}}(k)$. Given this metric, we now integrate eq. (2.21) over the manifold $\widetilde{Q}$ to define a $\mathbb{C}$-valued inner product of sections

$$\begin{aligned} \big\langle S, S' \big\rangle &= \frac{N_k}{\mathrm{Vol}_{\mathrm{CY}}(\widetilde{Q})} \int_{\widetilde{Q}} (S, S')(p)\, \mathrm{dVol}_{\mathrm{CY}} \\ &= \frac{N_k}{\mathrm{Vol}_{\mathrm{CY}}(\widetilde{Q})} \int_{\widetilde{Q}} \frac{S\,\bar{S}'}{\sum_{\alpha,\bar{\beta}} h^{\alpha\bar{\beta}} s_\alpha \bar{s}_{\bar{\beta}}}\, \mathrm{dVol}_{\mathrm{CY}}\,. \end{aligned} \tag{2.23}$$

Since $\langle\ ,\ \rangle$ is again sesquilinear, it is uniquely determined by its value on the basis sections s_α, that is, by the hermitian matrix

$$H_{\alpha\bar{\beta}} = \big\langle s_\alpha, s_\beta \big\rangle. \tag{2.24}$$

[3] In other words, at any given point one can only decide whether the section is zero or not zero.

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In general, the matrices $h^{\alpha\bar\beta}$ and $H_{\alpha\bar\beta}$ are completely different. However, for special metrics, they might coincide:

Definition. *Suppose that*

$$h^{\alpha\bar\beta} = \left(H_{\alpha\bar\beta}\right)^{-1}. \tag{2.25}$$

Then the metric h on the line bundle $\mathcal{O}_{\widetilde{Q}}(k)$ is called balanced.

We note that, in the balanced case, one can find a new basis of sections $\{\tilde s_\alpha\}_{\alpha=0}^{N_k-1}$ which simultaneously diagonalizes $\tilde H_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$ and $\tilde h^{\alpha\bar\beta} = \delta^{\alpha\bar\beta}$. The interesting thing about balanced metrics is that they have special curvature properties, in particular

Theorem 1 (Donaldson [30]) *For each $k \geq 1$ the balanced metric h exists and is unique. As $k \to \infty$, the sequence of metrics*

$$g^{(k)}_{i\bar\jmath} = \frac{1}{k\pi}\partial_i\bar\partial_{\bar\jmath} \ln \sum_{\alpha,\bar\beta=0}^{N_k-1} h^{\alpha\bar\beta} s_\alpha \bar s_{\bar\beta} \tag{2.26}$$

on $\widetilde{Q}$ converges to the unique Calabi-Yau metric for the given Kähler class and complex structure.

Hence, the problem of finding the Calabi-Yau metric boils down to finding the balanced metric for each k. Unfortunately, since $H_{\alpha\bar\beta}$ depends non-linearly on $h^{\alpha\bar\beta}$ one can not simply solve eq. (2.25) defining the balanced condition. However, iterating eq. (2.25) turns out to converge quickly. That is, let

$$T(h)_{\alpha\bar\beta} = H_{\alpha\bar\beta} = \frac{N_k}{\mathrm{Vol}_{\mathrm{CY}}\left(\widetilde{Q}\right)} \int_{\widetilde{Q}} \frac{s_\alpha \bar s_{\bar\beta}}{\sum_{\gamma\bar\delta} h^{\gamma\bar\delta} s_\gamma \bar s_{\bar\delta}} \,\mathrm{dVol}_{\mathrm{CY}} \tag{2.27}$$

be Donaldson's T-operator. Then

Theorem 2 (Donaldson, [21]) *For any initial metric h_0, the sequence*[4]

$$h_{n+1} = \big(T(h_n)\big)^{-1} \tag{2.28}$$

converges to the balanced metric as $n \to \infty$.

In practice, only very few (≤ 10) iterations are necessary to get very close to the fixed point. Henceforth, we will also refer to $g^{(k)}_{i\bar\jmath}$ in eq. (2.26), the approximating metric for fixed k, as a balanced metric.

[4] At this point, it is crucial to work with a *basis* of sections $s_0, \ldots, s_{N_k-1}$. For if there were a linear relation between them then the matrix $T(h)$ would be singular.

2.3 Integrating over the Calabi-Yau threefold

We still need to be able to integrate over the manifold in order to evaluate the T-operator. Luckily, we know the exact Calabi-Yau volume form,

$$\mathrm{dVol}_{\mathrm{CY}} = \Omega \wedge \bar{\Omega}, \tag{2.29}$$

since we can express the holomorphic volume form Ω as a Griffiths residue. To do this, first note that the hypersurface $\widetilde{Q} \subset \mathbb{P}^4$ has complex codimension one, so we can encircle any point in the transverse direction. The corresponding residue integral

$$\Omega = \oint \frac{\mathrm{d}^4 z}{\widetilde{Q}(z)} \tag{2.30}$$

is a nowhere vanishing holomorphic $(3,0)$-form and, hence, must be the holomorphic volume form Ω. As an example, consider the Fermat quintic defined by eq. (2.3). In a patch where we can use the homogeneous rescaling to set $z_0 = 1$ and where z_2, z_3, and z_4 are good local coordinates,

$$\Omega = \int \frac{\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_4}{1 + z_1^5 + z_2^5 + z_3^5 + z_4^5} = \frac{\mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}z_4}{5 z_1^4}. \tag{2.31}$$

To apply, for example, Simpson's rule to numerically integrate over the Calabi-Yau threefold one would need local coordinate charts. However, there is one integration scheme that avoids having to go into these details: approximate the integral by N_p random points $\{p_i\}$,

$$\frac{1}{N_p} \sum_{i=1}^{N_p} f(p_i) \longrightarrow \int f \,\mathrm{dVol}\,. \tag{2.32}$$

Of course, we have to define which "random" distribution the points lie on, which in turn determines the integration measure dVol. In practice, we will only be able to generate points with the *wrong* random distribution, leading to some auxiliary distribution $\mathrm{d}A$. However, one can trivially account for this by weighting the points with $w_i = \Omega \wedge \bar{\Omega} / \mathrm{d}A$,

$$\frac{1}{N_p} \sum_{i=1}^{N_p} f(p_i) w_i = \frac{1}{N_p} \sum_{i=1}^{N_p} f(p_i) \frac{\Omega \wedge \bar{\Omega}}{\mathrm{d}A} \longrightarrow \int f \; \frac{\Omega \wedge \bar{\Omega}}{\mathrm{d}A} \,\mathrm{d}A = \int f \;\, \mathrm{dVol}_{\mathrm{CY}}\,. \tag{2.33}$$

Note that taking $f = 1$ implies

$$\frac{1}{N_p} \sum_{i=1}^{N_p} w_i = \mathrm{Vol}_{\mathrm{CY}}(\widetilde{Q}). \tag{2.34}$$

Points from patches. We start out with what is probably the most straightforward way to pick random points. This method only works on the Fermat quintic, to which we now restrict. Let us split $\mathbb{P}^4$ into $5 \cdot 4 = 20$ closed sets

$$U_{\ell m} = \Big\{ \big[z_0 : z_1 : z_2 : z_3 : z_4 \big] \Big| \tag{2.35}$$

$$|z_\ell| = \max(|z_0|, \ldots, |z_4|),\ |z_m| = \max(|z_0|, \ldots, \widehat{|z_\ell|}, \ldots, |z_4|) \Big\}. \tag{2.36}$$

In other words, z_ℓ has the largest absolute value and z_m has the second-largest absolute value. They intersect in real codimension-1 boundaries where the absolute values are the same and induce the decomposition

$$\widetilde{Q}_F = \bigcup_{\ell,m} \widetilde{Q}_{F,\ell m} \tag{2.37}$$

with $\widetilde{Q}_{F,\ell m} = \widetilde{Q}_F \cap U_{\ell m}$. Since permuting coordinates is a symmetry of the Fermat quintic, it suffices to consider $\widetilde{Q}_{F,01}$. We define "random" points by

- Pick x, y, $z \in \mathbb{C}_{\leq 1}$ on the complex unit disk with the standard "flat" distribution.
- Test whether
$$|x|, |y|, |z| \leq \left|1 + x^5 + y^5 + z^5\right|^{\frac{1}{5}} \leq 1. \tag{2.38}$$
  If this is not satisfied, start over and pick new x, y, and z. Eventually, the above inequality will be satisfied.
- The "random" point is now
$$\left[1 : -\left(1 + x^5 + y^5 + z^5\right)^{\frac{1}{5}} : x : y : z\right] \in \widetilde{Q}_{F,01}, \tag{2.39}$$
  where one chooses a uniformly random phase for the fifth root of unity.

By construction, the auxiliary measure is then independent of the position $(x, y, z) \in \widetilde{Q}_{F,\ell m}$. Hence,

$$\mathrm{d}A = \frac{1}{20} \mathrm{d}^2x \wedge \mathrm{d}^2y \wedge \mathrm{d}^2z. \tag{2.40}$$

Points from intersecting lines with the quintic. The previous definition only works on the Fermat quintic, but not on arbitrary quintics. A much better algorithm [23] is to pick random lines

$$L \simeq \mathbb{P}^1 \subset \mathbb{P}^4 . \tag{2.41}$$

Any line L determines 5 points by the intersection $L \cap \widetilde{Q} = \{5\text{pt.}\}$ whose coordinates can be found by solving a quintic polynomial (in one variable) numerically. Explicitly, a line can be defined by two distinct points

$$p = [p_0 : p_1 : p_2 : p_3 : p_4],\ q = [q_0 : q_1 : q_2 : q_3 : q_4]\ \in \mathbb{P}^4 \tag{2.42}$$

as

$$L : \mathbb{C} \cup \{\infty\} \to \mathbb{P}^4,\ t \mapsto [p_0 + tq_0 : p_1 + tq_1 : p_2 + tq_2 : p_3 + tq_3 : p_4 + tq_4]. \tag{2.43}$$

The 5 intersection points $L \cap \widetilde{Q}$ are then given by the 5 solutions of

$$\widetilde{Q} \circ L(t) = \widetilde{Q}\big(p_0 + tq_0, p_1 + tq_1, p_2 + tq_2, p_3 + tq_3, p_4 + tq_4\big) = 0. \tag{2.44}$$

Clearly, the auxiliary measure will depend on how we pick "random" lines. The easiest way is to choose lines uniformly distributed with respect to the SU(5) action on $\mathbb{P}^4$. Note

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that a line L is Poincaré dual to a $(3,3)$-current, that is, a $(3,3)$-form whose coefficients are delta-functions supported on the line L. For the expected distribution of lines, we then average over all "random" configurations of lines. Because of this averaging procedure, the Poincaré dual[5] of the expected distribution of lines $\langle L \rangle$ is a smooth $(3,3)$ form. Since there is (up to scale) only one SU(5)-invariant $(3,3)$ form on $\mathbb{P}^4$, the expected distribution of lines must be

$$\langle L \rangle \sim \omega_{\mathrm{FS}}^3, \tag{2.45}$$

where ω_{FS} is the Kähler form defined by the unique SU(5)-invariant Fubini-Study Kähler potential eq. (2.9). Restricting both sides to an embedded quintic $i : \widetilde{Q} \hookrightarrow \mathbb{P}^4$, we obtain the auxiliary measure as the expected distribution of the intersection points,

$$\mathrm{d}A = \langle \widetilde{Q} \cap L \rangle \sim i^*\big(\omega_{\mathrm{FS}}^3\big). \tag{2.46}$$

As a final remark, note that the the symmetry of the ambient space is, in general, not enough to unambiguously determine the auxiliary measure. It is, as we just saw, sufficient for any quintic hypersurface $\widetilde{Q}$. However, for more complicated threefolds one needs a more general theory. We will have to come back to this point in section 5.4.

2.4 Results

Following the algorithm laid out in this section, we can now compute the successive approximations to the Calabi-Yau metric on $\widetilde{Q}$. In order to test the result, we need some kind of measure for how close the approximate metric is to the Calabi-Yau metric. Douglas et al. [23] proposed the following: First, remember that the Kähler form ω eq. (2.6) is the Calabi-Yau Kähler form if and only if its associated volume form ω^3 is proportional to the Calabi-Yau volume form eq. (2.30). That is,

$$\omega^3(p) = (\text{const.}) \times \Big(\Omega(p) \wedge \bar{\Omega}(p)\Big) \quad \forall p \in \widetilde{Q} \tag{2.47}$$

(the Monge-Ampére equation) with a non-vanishing proportionality constant independent of $p \in \widetilde{Q}$.[6] Let us define

$$\mathrm{Vol}_{\mathrm{K}}\big(\widetilde{Q}\big) = \int_{\widetilde{Q}} \omega^3 \tag{2.48a}$$

and recall that

$$\mathrm{Vol}_{\mathrm{CY}}\big(\widetilde{Q}\big) = \int_{\widetilde{Q}} \Omega \wedge \bar{\Omega}. \tag{2.48b}$$

The ratio of these two constants determines the proportionality factor in eq. (2.47). This equation can now be rewritten

$$\frac{\omega^3(p)}{\mathrm{Vol}_{\mathrm{K}}\big(\widetilde{Q}\big)} = \frac{\Omega(p) \wedge \bar{\Omega}(p)}{\mathrm{Vol}_{\mathrm{CY}}\big(\widetilde{Q}\big)} \qquad \forall p \in \widetilde{Q}. \tag{2.49}$$

[5] By the usual abuse of notation, we will not distinguish Poincaré dual quantities in the following.

[6] And varying over the moduli space. However, this will not concern us here.

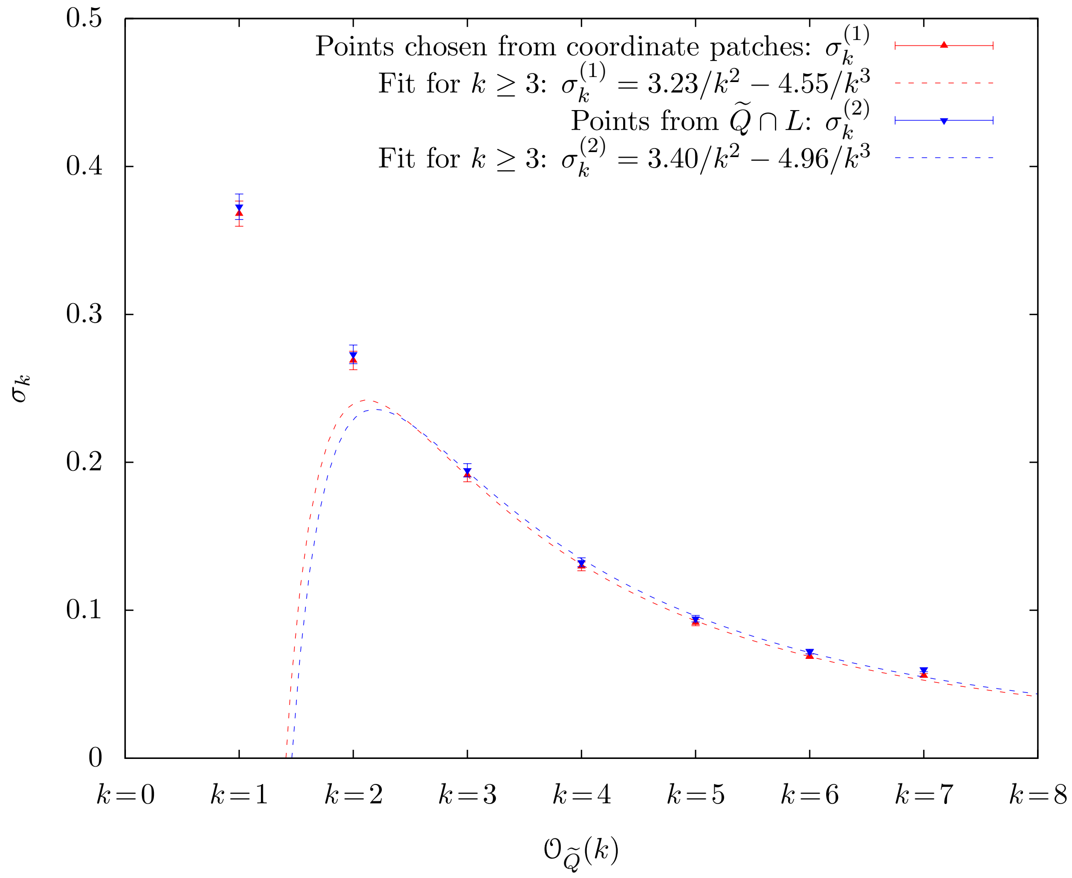

Figure 1: The error measure σ_k for the metric on the Fermat quintic, computed with the two different point generation algorithms described in section 2.3. In each case we iterated the T-operator 10 times, numerically integrating over $N_p = 200{,}000$ points. Then we evaluated σ_k using 10,000 different test points. The error bars are the numerical errors in the σ_k integral.

Note that one often demands that the two constants, eqs. (2.48a) and (2.48b), are unity by rescaling ω and Ω respectively. However, this would be cumbersome later on and we will not impose this normalization. Then the integral

$$\sigma(\widetilde{Q}) = \frac{1}{\mathrm{Vol}_{\mathrm{CY}}\left(\widetilde{Q}\right)} \int_{\widetilde{Q}} \left| 1 - \frac{\omega^3 \Big/ \mathrm{Vol}_{\mathrm{K}}\left(\widetilde{Q}\right)}{\Omega \wedge \bar{\Omega} \Big/ \mathrm{Vol}_{\mathrm{CY}}\left(\widetilde{Q}\right)} \right| \mathrm{dVol}_{\mathrm{CY}} \tag{2.50}$$

vanishes if and only if ω is the Calabi-Yau Kähler form. In practice, Donaldson's algorithm determines successive approximations to the Calabi-Yau metric. Since we know the exact Calabi-Yau volume form $\Omega \wedge \bar{\Omega}$, only ω is approximate and depends on the degree k. We define σ_k to be the above integral evaluated with this degree-k approximation to the Calabi-Yau Kähler form.

Let us quickly summarize the steps necessary to compute the metric. To do that, one has to

1. Choose a degree k at which to compute the balanced metric which will approximate the Calabi-Yau metric.

2. Choose the number N_p of points, and generate this many points $\{p_i\}_{i=1}^{N_p}$ on $\widetilde{Q}$. Although k and N_p can be chosen independently, we will argue below that N_p should be sufficiently larger than N_k^2 for accuracy.

3. For each point p_i, compute its weight $w_i = \mathrm{d}A(p_i)/\left(\Omega \wedge \bar{\Omega}\right)$.

4. Calculate a basis $\{s_\alpha\}_{\alpha=0}^{N_k-1}$ for the quotient eq. (2.15) at degree k.

5. At each point p_i, calculate the (complex) numbers $\{s_\alpha(p_i)\}_{\alpha=0}^{N_k-1}$ and, hence, the integrand of the T-operator.

6. Choose an initial invertible, hermitian matrix for $h^{\gamma\bar{\delta}}$. Now perform the numerical integration

$$T(h)_{\alpha\bar{\beta}} = \frac{N_k}{\sum_{j=1}^{N_p} w_j} \sum_{i=1}^{N_p} \frac{s_\alpha(p_i)\, \overline{s_\beta(p_i)} w_i}{\sum_{\gamma\bar{\delta}} h^{\gamma\bar{\delta}}\, s_\gamma(p_i)\, \overline{s_\delta(p_i)}}. \tag{2.51}$$

7. Set the new $h^{\alpha\bar{\beta}}$ to be $h^{\alpha\bar{\beta}} = \left(T_{\alpha\bar{\beta}}\right)^{-1}$.

8. Return to point 6 and repeat until $h^{\alpha\bar{\beta}}$ converges close to its fixed point. In practice, this procedure is insensitive to the initial choice of $h^{\alpha\bar{\beta}}$ and fewer than 10 iterations suffice.

Having determined the balanced $h^{\alpha\bar{\beta}}$, we can evaluate the metric $g_{i\bar{j}}^{(k)}$ using eq. (2.26) and, hence, the Kähler form $\omega(p)$ at each point p, see eq. (2.6). Now form $\omega^3(p)$. This lets us compute σ_k by the following steps:

1. The σ_k integral requires much less accuracy, so one may pick a smaller number N_p of points $\{p_i\}_{i=1}^{N_p}$.

2. Compute

$$\mathrm{Vol}_{\mathrm{CY}} = \frac{1}{N_p}\sum_{i=1}^{N_p} w_i, \qquad \mathrm{Vol}_{\mathrm{K}} = \frac{1}{N_p}\sum_{i=1}^{N_p} \frac{\omega^3(p_i)}{\Omega(p_i) \wedge \overline{\Omega(p_i)}} w_i, \tag{2.52}$$

   which numerically approximate $\int_{\widetilde{Q}} \Omega \wedge \bar{\Omega}$ and $\int_{\widetilde{Q}} \omega^3$, respectively.

3. The numerical integral approximating σ_k is

$$\sigma_k = \frac{1}{N_p\, \mathrm{Vol}_{\mathrm{CY}}} \sum_{i=1}^{N_p} \left| 1 - \frac{\omega(p_i)^3 / \mathrm{Vol}_{\mathrm{K}}}{\Omega(p_i) \wedge \overline{\Omega(p_i)} / \mathrm{Vol}_{\mathrm{CY}}} \right| w_i. \tag{2.53}$$

As a first application, we apply this procedure to compute the Calabi-Yau metric for the simple Fermat quintic $\widetilde{Q}_F$ defined by eq. (2.3). In this case, there are two point selection algorithms, both given in section 2.3. We do the calculation for each and show the results in figure 1. One can immediately see that both point selection strategies give the same result, as they should. In fact, there is a theoretical prediction for how fast σ_k converges to 0, see [23, 25, 30]. Expanding in $\frac{1}{k}$, the error goes to zero at least as fast as

$$\sigma_k = \frac{S_2}{k^2} + \frac{S_3}{k^3} + \cdots , \qquad S_i \in \mathbb{R}. \tag{2.54}$$

In particular, the coefficient of $\frac{1}{k}$ is proportional to the scalar curvature and vanishes on a Calabi-Yau manifold. In figure 1, we fit $\sigma_k = \frac{S_2}{k^2} + \frac{S_3}{k^3}$ for $k \geq 3$ and find good agreement with the data points.

An important question is how many points are necessary to approximate the Calabi-Yau threefold in the numerical integration for any given k. The problem is that we really are trying to compute the $N_k \times N_k$-matrix $h^{\alpha\bar{\beta}}$, whose dimension increases quickly with k, see table 1. Hence, to have more equations than indeterminates, we expect to need

$$N_p > N_k^2 \tag{2.55}$$

points to evaluate the integrand of the T-operator on. To numerically test this, we compute σ_k using different numbers of points N_p. The result is displayed in figure 2, where we used the more convenient logarithmic scale for σ_k. Clearly, the error measure σ_k starts out decreasing with k. However, at some N_p-dependent point it reaches a minimum and then starts to increase. In figure 3, we plot the same σ_k as a function of N_k^2. This confirms our guess that we need $N_p > N_k^2$ points in order to accurately perform the numerical integration. One notes that the data points in figure 2 seem to approach a straight line as we increase N_p. This would suggest an exponential fall-off

$$\sigma_k \approx 0.523 e^{-0.324k}. \tag{2.56}$$

It is possible, therefore, that the theoretical error estimate eq. (2.54) could be improved upon.

So far, we have applied our procedure to the Fermat quintic $\widetilde{Q}_F$ for simplicity. However, our formalism applies equally well to any quintic $\widetilde{Q}$ in the 101-dimensional complex structure moduli space with the proviso that, for a non-Fermat quintic, one must use the $L \cap \widetilde{Q}$ method of choosing points. An important property of the programs that implement our procedure is that they make no assumptions about the form of the quintic polynomial eq. (2.2). We proceed as follows. First, fix a quintic by randomly (in the usual flat distribution) choosing each coefficient $c_{(n_0,n_1,n_2,n_3,n_4)}$ on the unit disk, see eq. (2.2). Then approximate the Calabi-Yau metric via Donaldson's algorithm and compute the error measure σ_k. In figure 4, we present the results for σ_k for five randomly chosen quintics, and compare them to the Fermat quintic. We observe that the convergence to the Calabi-Yau metric does not strongly depend on the complex structure parameters.

3. Group actions and invariants

3.1 Quotients and covering spaces

Thus far, we have restricted our formalism to quintic Calabi-Yau threefolds $\widetilde{Q} \subset \mathbb{P}^4$. These are, by construction, simply connected. However, for applications in heterotic string theory

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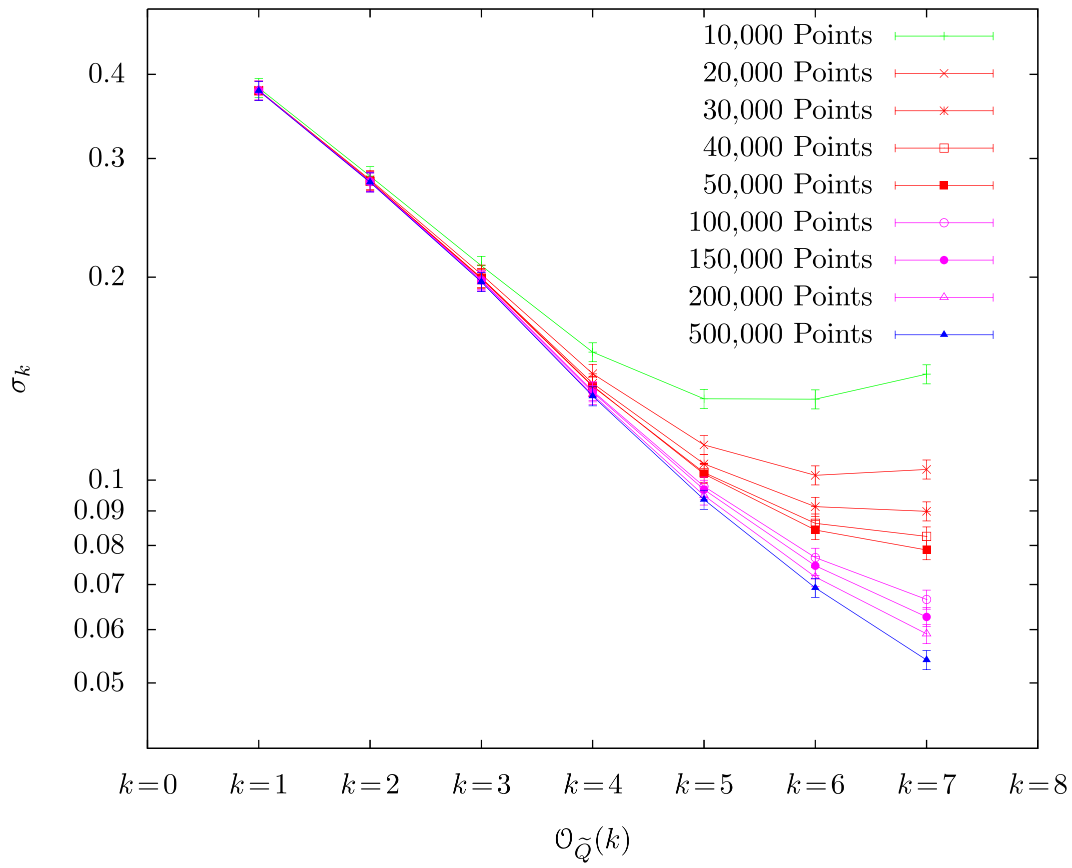

Figure 2: The error measure σ_k for the balanced metric on the Fermat quintic as a function of k, computed by numerical integration with different numbers of points N_p. In each case, we iterated the T-operator 10 times and evaluated σ_k on 5,000 different test points. Note that we use a logarithmic scale for the σ_k axis.

we are particularly interested in non-simply connected Calabi-Yau manifolds where one can reduce the number of quark/lepton generations and turn on discrete Wilson lines [31 – 37]. Therefore, it is of obvious interest to compute the metrics in such cases. However, these manifolds are more complicated than hypersurfaces in projective spaces. In fact, any complete intersection in a smooth toric variety will be simply connected.[7] Therefore, we are usually forced to study non-simply connected Calabi-Yau threefolds Y,

$$\pi_1(Y) = \Pi \neq 1, \tag{3.1}$$

via their universal covering space $\widetilde{Y}$ and the free group action $\Pi : \widetilde{Y} \to \widetilde{Y}$.

In order to carry through Donaldson's algorithm on Y, we now need to generalize the notion of "homogeneous polynomials" to arbitrary varieties. As mentioned previously, the

[7]Note, however, that there are 16 cases of smooth, non-simply connected hypersurfaces in singular toric varieties [38].

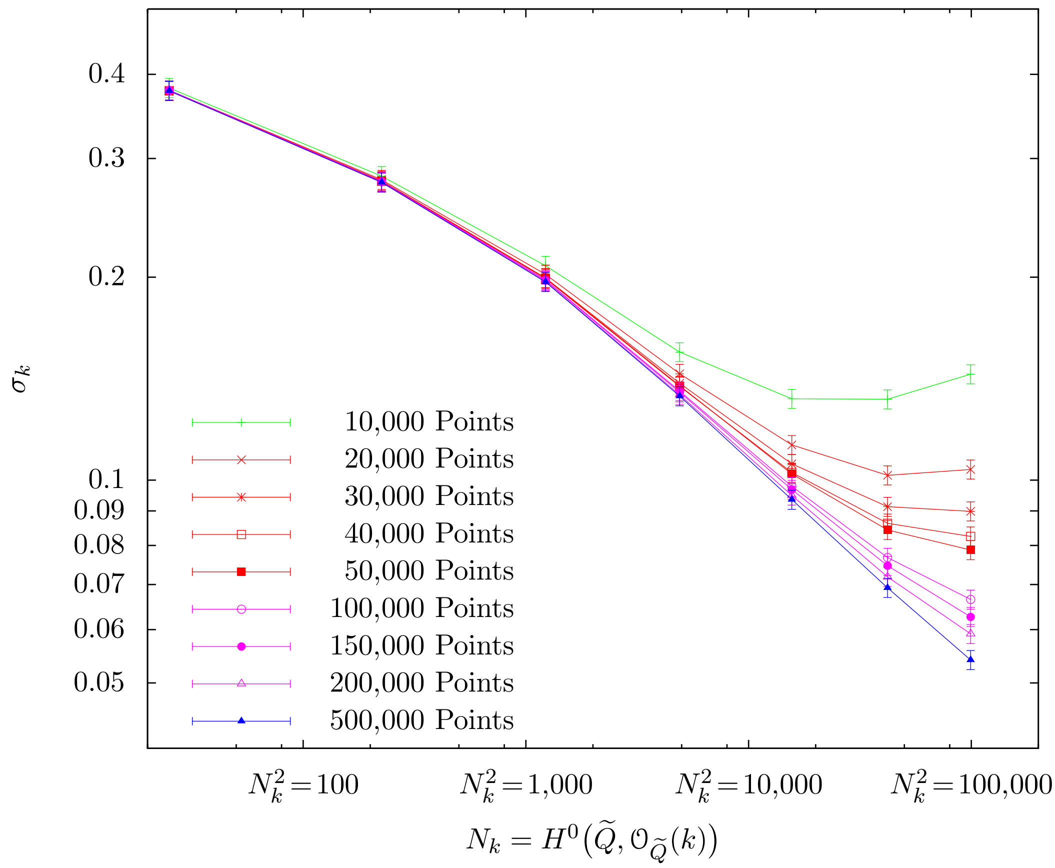

Figure 3: The error measure σ_k for the balanced metric on the Fermat quintic as a function of N_k^2 = number of entries in $h^{\alpha\bar{\beta}} \in \mathrm{Mat}_{N_k \times N_k}$. In other words, evaluating the T-operator requires N_k^2 scalar integrals. In each case, we iterated the T-operator 10 times and finally evaluated σ_k using 5,000 different test points. We use a logarithmic scale for both axes.

homogeneous coordinates on the quintic $\widetilde{Q} \in \mathbb{P}^4$ can be interpreted as the basis of sections of the line bundle $\mathcal{O}_{\widetilde{Q}}(1)$,

$$\mathrm{span}\{z_0, z_1, z_2, z_3, z_4\} = H^0\left(\widetilde{Q}, \mathcal{O}_{\widetilde{Q}}(1)\right). \tag{3.2}$$

The special property of $\mathcal{O}_{\widetilde{Q}}(1)$ is that it is "very ample", that is, its sections define an embedding

$$\Phi_{\mathcal{O}_{\widetilde{Q}}(1)} : \widetilde{Q} \to \mathbb{P}^4,\ x \mapsto \big[z_0(x) : z_1(x) : z_2(x) : z_3(x) : z_4(x)\big]. \tag{3.3}$$

Hence, we need to pick a "very ample" line bundle on $\widetilde{Y}$ in order to compute the metric there. Furthermore, to discuss Y, we will also need to "mod out" by the group action. It follows that the group Π must act properly on the line bundle. In mathematical terms this

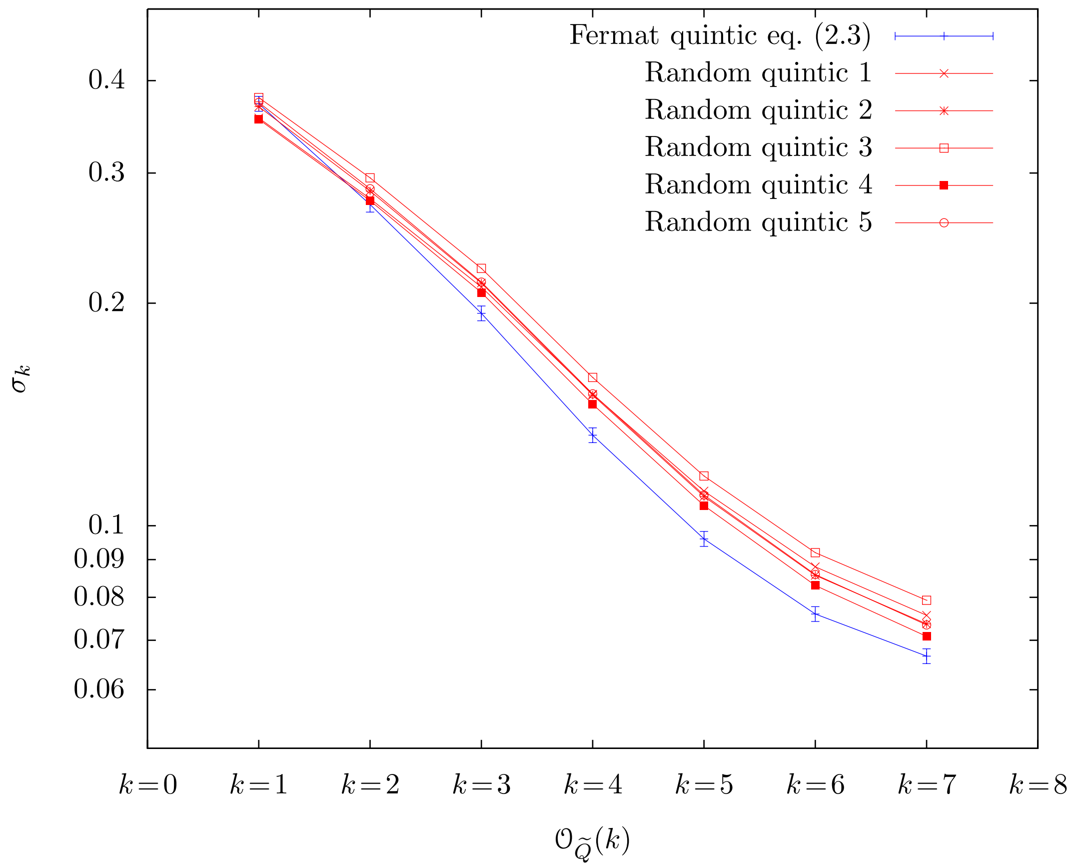

Figure 4: The error measure σ_k as a function of k for five random quintics, as well as for the Fermat quintic. The random quintics are the sum over the 126 quintic monomials in 5 homogeneous variables with coefficients random on the unit disk. We use a logarithmic scale for σ_k.

is called an "equivariant line bundle", and there is a one-to-one correspondence

$$\boxed{\begin{array}{c}\Pi\text{-equivariant}\\ \text{line bundles on } \widetilde{Y}\end{array}} \ \underset{\Pi^*}{\overset{\Pi}{\rightleftarrows}}\ \boxed{\begin{array}{c}\text{Line bundles}\\ \text{on } Y.\end{array}} \tag{3.4}$$

Let us denote such a line bundle on $\widetilde{Y}$ by $\mathcal{L}$. We are specifically interested in the sections of this line bundle, since they generalize the homogeneous coordinates. The important observation here is that the sections of a Π-equivariant line bundle on $\widetilde{Y}$ themselves form a representation of Π. Furthermore, the Π-invariant sections correspond to the sections on the quotient. That is,

$$H^0\big(\widetilde{Y}, \mathcal{L}\big)^{\Pi} = H^0\big(Y, \mathcal{L}/\Pi\big). \tag{3.5}$$

Hence, in order to compute the metric on the quotient $Y = \widetilde{Y}/\Pi$, we can work on the covering space $\widetilde{Y}$ if we simply replace all sections by the Π-invariant sections.

In this paper, we will always consider the case where $\widetilde{Y}$ is a hypersurface or a complete intersection in (products of) projective spaces. Then

- The sections on the ambient projective space are homogeneous polynomials.
- The sections on $\widetilde{Y}$ are the quotient of these polynomials by the defining equations.
- The invariant sections on $\widetilde{Y}$ are the invariant homogeneous polynomials modulo the invariant polynomials generated by the defining equations.

The mathematical framework for counting and finding these invariants is provided by Invariant Theory [29], which we review in the remainder of this section.

3.2 Poincaré and Molien

Let $\mathbb{C}[\vec{x}]$ be a polynomial ring in n commuting variables

$$\vec{x} = \big(x_1, \dots, x_n\big). \tag{3.6}$$

As a vector space over the ground field $\mathbb{C}$, it is generated by all monomials

$$\mathbb{C}[\vec{x}] = \mathbb{C}1 \oplus \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n \oplus \mathbb{C}x_1^2 \oplus \cdots . \tag{3.7}$$

Clearly, $\mathbb{C}[\vec{x}]$ is an infinite dimensional vector space. However, at each degree k we have a finite dimensional vector space of homogeneous degree-k polynomials. A concrete basis for the degree-k polynomials would be all distinct monomials of that degree.

By definition, the Poincaré series is the generating function for the dimensions of the vector subspaces of fixed degree, that is,

$$P\big(\mathbb{C}[\vec{x}], t\big) = \sum_{k=0}^{\infty} \Big(\dim_{\mathbb{C}} \mathbb{C}[\vec{x}]_k\Big)\, t^k \tag{3.8}$$

where $\mathbb{C}[\vec{x}]_k$ is the vector subspace of $\mathbb{C}[\vec{x}]$ of degree k. The monomials of the polynomial ring in n commuting variables x_1, ..., x_n can be counted just like n species of bosons, and one obtains

$$P\big(\mathbb{C}[\vec{x}], t\big) = \prod_n \frac{1}{1-t} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} t^k. \tag{3.9}$$

We have already mentioned that the homogeneous degree-k polynomials in n variables are just the sections of $\mathcal{O}_{\mathbb{P}^{n-1}}(k)$. Hence, the number of degree-k polynomials is the same as the dimension of the space of sections of the line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(k)$,

$$\dim_{\mathbb{C}} \mathbb{C}[\vec{x}]_k = h^0\big(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(k)\big). \tag{3.10}$$

Acknowledging this geometric interpretation, we also write

$$P\big(\mathcal{O}_{\mathbb{P}^{n-1}}, t\big) = \sum_{k=0}^{\infty} h^0\big(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(k)\big)\, t^k = P\big(\mathbb{C}[\vec{x}], t\big). \tag{3.11}$$

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Furthermore, note that

$$P(M \oplus M', t) = P(M, t) + P(M', t) \tag{3.12}$$

for any rings M and M'.

A n-dimensional representation of a finite group G generates a group action on the polynomials eq. (3.7). One is often interested in the invariant polynomials under this group action, which again form a ring $\mathbb{C}[\vec{x}]^G$. Clearly, the invariant ring is a subring of $\mathbb{C}[\vec{x}]$. Since the group action preserves the degree of a polynomial, one can again define the Poincaré series of the invariant ring,

$$P\big(\mathbb{C}[\vec{x}]^G, t\big) = \sum_{k=0}^{\infty} \Big(\dim_{\mathbb{C}} \mathbb{C}[\vec{x}]_k^G\Big)\, t^k. \tag{3.13}$$

The coefficients in eq. (3.13) can be obtained using

Theorem 3 (Molien) *Let $G \subset GL(n, \mathbb{C})$ be a finite matrix group acting linearly on the n variables $\vec{x} = (x_1, \ldots, x_n)$. Then the Poincaré series of the ring of invariant polynomials, that is, the generating function for the number of invariant polynomials of each degree, is given by*

$$P\left(\mathbb{C}[\vec{x}]^G, t\right) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - gt)}. \tag{3.14}$$

Equation (3.14) is called the Molien formula.

3.3 Hironaka decomposition

Although eq. (3.14) contains important information about $\mathbb{C}[x_1, \ldots, x_n]^G$, the most detailed description is provided by the Hironaka decomposition, which we discus next. To construct this, one first needs to find n homogeneous polynomials $\theta_1, \ldots, \theta_n$, invariant under the group action, such that the quotient

$$\mathbb{C}[x_1, \ldots, x_n] / \langle \theta_1, \ldots, \theta_n \rangle \tag{3.15}$$

is zero-dimensional. The above condition is equivalent [39] to demanding that the system $\theta_i = 0$, $i = 1, \ldots n$ has only the trivial solution. This guaranties that the θ_i are algebraically independent. Then

Theorem 4 (Hironaka decomposition) *With respect to θ_1, …, θ_n chosen as above, the ring of G-invariant polynomials can be decomposed as*

$$\mathbb{C}[\vec{x}]^G = \eta_1 \mathbb{C}[\theta_1, \ldots, \theta_n] \oplus \eta_2 \mathbb{C}[\theta_1, \ldots, \theta_n] \oplus \cdots \oplus \eta_s \mathbb{C}[\theta_1, \ldots, \theta_n]. \tag{3.16}$$

Clearly, the η_i are themselves G-invariant polynomials in $\mathbb{C}[\vec{x}]$. Thus any G-invariant polynomial is a unique linear combination of η_i's, where the coefficients are polynomials in θ_i. The polynomials θ_i are called the "primary" invariants and η_j the "secondary" invariants. Note that, while the number of primary invariants is fixed by the number of variables x_1, …, x_n, the number s of secondary polynomials depends on our choice of primary invariants. Using eq. (3.9) with each x_i replaced by θ_i, we find that the Poincaré series for $\mathbb{C}[\theta_1, \ldots, \theta_n]$ is given by

$$P\big(\mathbb{C}[\theta_1, \ldots, \theta_n], t\big) = \frac{1}{\big(1 - t^{\deg(\theta_1)}\big) \ldots \big(1 - t^{\deg(\theta_n)}\big)}. \tag{3.17}$$

Moreover, multiplication by η_i shifts all degrees by $\deg(\eta_i)$. Therefore, applying eq. (3.12) we obtain the Poincaré series for the Hironaka decomposition,

$$\begin{aligned} P\big(\mathbb{C}[\vec{x}]^G, t\big) &= \frac{t^{D_1}}{(1-t^{d_1}) \cdots (1-t^{d_n})} + \cdots + \frac{t^{D_s}}{(1-t^{d_1}) \cdots (1-t^{d_n})} \\ &= \frac{t^{D_1} + \cdots + t^{D_s}}{(1-t^{d_1}) \cdots (1-t^{d_n})}, \end{aligned} \tag{3.18}$$

where $D_j = \deg(\eta_j)$ and $d_i = \deg(\theta_i)$. Each term in the numerator of eq. (3.18) corresponds to a secondary invariant.

4. Four-generation quotient of quintics

4.1 Four generation models

Were one to compactify the heterotic string on a generic quintic $\widetilde{Q}$ using the standard embedding, then the four-dimensional effective theory would contain $\frac{1}{2}\chi(\widetilde{Q}) = 100$ net generations. A well known way to reduce this number [1] is to compactify on quintics that admit a fixed point free $\mathbb{Z}_5 \times \mathbb{Z}_5$ action. In that case, the quotient manifold $Q = \widetilde{Q}\big/(\mathbb{Z}_5 \times \mathbb{Z}_5)$ has only $\frac{1}{2}\chi(Q) = \frac{100}{|\mathbb{Z}_5 \times \mathbb{Z}_5|} = 4$ generations. In this section, these special quintics and their $\mathbb{Z}_5 \times \mathbb{Z}_5$ quotient will be described. We then compute the Calabi-Yau metrics directly on these quotients Q using a generalization of our previous formalism.

Recall from section 2 that a generic quintic $\widetilde{Q} \subset \mathbb{P}^4$ is defined as the zero locus of a degree-5 polynomial of the form eq. (2.2). In general, it is the sum of 126 degree-5 monomials, leading to 126 coefficients $c_{(n_0,n_1,n_2,n_3,n_4)} \in \mathbb{C}$. However, not all of these quintic threefolds admit a fixed point free $\mathbb{Z}_5 \times \mathbb{Z}_5$ action. To be explicit, we will consider the following two actions on the five homogeneous variables defining $\mathbb{P}^4$,

$$\begin{aligned} g_1 \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} &= \begin{pmatrix} 0&0&0&0&1 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \\ g_2 \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} &= \begin{pmatrix} 1&0&0&0&0 \\ 0&e^{\frac{2\pi i}{5}}&0&0&0 \\ 0&0&e^{2\frac{2\pi i}{5}}&0&0 \\ 0&0&0&e^{3\frac{2\pi i}{5}}&0 \\ 0&0&0&0&e^{4\frac{2\pi i}{5}} \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}. \end{aligned} \tag{4.1}$$

Clearly $g_1^5 = 1 = g_2^5$, but they do not quite commute:

$$g_1 g_2 = e^{\frac{2\pi i}{5}} g_2 g_1 \quad \Leftrightarrow \quad g_1 g_2 g_1^{-1} g_2^{-1} = e^{\frac{2\pi i}{5}}. \tag{4.2}$$

However, even though g_1 and g_2 do not form a matrix representation of $\mathbb{Z}_5 \times \mathbb{Z}_5$, they do generate a $\mathbb{Z}_5 \times \mathbb{Z}_5$ action on $\mathbb{P}^4$ because on the level of homogeneous coordinates we have to identify

$$\begin{aligned} [z_0 : z_1 : z_2 : z_3 : z_4] &= \left[e^{\frac{2\pi i}{3}} z_0 : e^{\frac{2\pi i}{3}} z_1 : e^{\frac{2\pi i}{3}} z_2 : e^{\frac{2\pi i}{3}} z_3 : e^{\frac{2\pi i}{3}} z_4\right] \\ &= g_1 g_2 g_1^{-1} g_2^{-1}\Big([z_0 : z_1 : z_2 : z_3 : z_4]\Big). \end{aligned} \tag{4.3}$$

If the quintic polynomial $\widetilde{Q}(z)$ is $\mathbb{Z}_5 \times \mathbb{Z}_5$-invariant, then the corresponding hypersurface will inherit this group action. One can easily verify that the dimension of the space of invariant homogeneous degree-5 polynomials is 6, as we will prove in eq. (4.18) below. Taking into account that one can always multiply the defining equation by a constant, there are 5 independent parameters $\phi_1, \ldots, \phi_5 \in \mathbb{C}$. Thus the $\mathbb{Z}_5 \times \mathbb{Z}_5$ symmetric quintics form a five parameter family which, at a generic point in the moduli space, can be written as

$$\begin{aligned} \widetilde{Q}(z) = &\left(z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5\right) \\ &+ \phi_1\left(z_0 z_1 z_2 z_3 z_4\right) \\ &+ \phi_2\left(z_0^3 z_1 z_4 + z_0 z_1^3 z_2 + z_0 z_3 z_4^3 + z_1 z_2^3 z_3 + z_2 z_3^3 z_4\right) \\ &+ \phi_3\left(z_0^2 z_1 z_2^2 + z_1^2 z_2 z_3^2 + z_2^2 z_3 z_4^2 + z_3^2 z_4 z_0^2 + z_4^2 z_0 z_1^2\right) \\ &+ \phi_4\left(z_0^2 z_1^2 z_3 + z_1^2 z_2^2 z_4 + z_2^2 z_3^2 z_0 + z_3^2 z_4^2 z_1 + z_4^2 z_0^2 z_2\right) \\ &+ \phi_5\left(z_0^3 z_2 z_3 + z_1^3 z_3 z_4 + z_2^3 z_4 z_0 + z_3^3 z_0 z_1 + z_4^3 z_1 z_2\right). \end{aligned} \tag{4.4}$$

The explicit form of these invariant polynomials is derived in section 4.3 and given in eq. (4.18). Note that, even though the $\mathbb{Z}_5 \times \mathbb{Z}_5$ action on $\mathbb{P}^4$ necessarily has fixed points, one can check that a generic (that is, for generic $\phi_1, \ldots, \phi_5$) quintic threefold $\widetilde{Q}$ is fixed-point free.

Now choose any quintic defined by eq. (4.4). Since the $\mathbb{Z}_5 \times \mathbb{Z}_5$ action on it is fixed point free, the quotient

$$Q = \widetilde{Q}\Big/\left(\mathbb{Z}_5 \times \mathbb{Z}_5\right) \tag{4.5}$$

is again a smooth Calabi-Yau threefold. Its Hodge diamond is given by [40]

$$h^{p,q}(Q) = h^{p,q}\Big(\widetilde{Q}/\left(\mathbb{Z}_5 \times \mathbb{Z}_5\right)\Big) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 5 & & 5 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}, \tag{4.6}$$

where we again see that there is a $h^{2,1}(Q) = 5$-dimensional complex structure moduli space parametrized by the coefficients $\phi_1, \ldots, \phi_5$.

4.2 Sections on the quotient

We now extend Donaldson's algorithm to compute the Calabi-Yau metric directly on the quotient $Q = \widetilde{Q}/(\mathbb{Z}_5 \times \mathbb{Z}_5)$. To do this, we will need to count and then explicitly construct the $\mathbb{Z}_5 \times \mathbb{Z}_5$ invariant sections, that is, the $\mathbb{Z}_5 \times \mathbb{Z}_5$ invariant polynomials, on the covering space $\widetilde{Q} \in \mathbb{P}^4$, as discussed in section 3.1. These then descend to the quotient Q and can be used to parametrize the Kahler potential and the approximating balanced metrics.

One technical problem, however, is that the two group generators g_1 and g_2 in eq. (4.1) do not commute; they only commute up to a phase. Therefore, the homogeneous coordinates

$$\operatorname{span}\left\{z_0, z_1, z_2, z_3, z_4\right\} = H^0\left(\widetilde{Q}, \mathcal{O}_{\widetilde{Q}}(1)\right) \tag{4.7}$$

do not carry a $\mathbb{Z}_5 \times \mathbb{Z}_5$ representation. The solution to this problem is to enlarge the group. Each generator has order 5 and, even though they do not quite generate $\mathbb{Z}_5 \times \mathbb{Z}_5$, they commute up to a phase. Hence, g_1 and g_2 generate the "central extension"

$$1 \longrightarrow \mathbb{Z}_5 \longrightarrow G \longrightarrow \mathbb{Z}_5 \times \mathbb{Z}_5 \longrightarrow 1 \tag{4.8}$$

with $|G| = 125$ elements. This group G is also called a Heisenberg group since it is formally the same as $[x, p] = 1$, only in this case over $\mathbb{Z}_5$. It follows that $H^0(\widetilde{Q}, \mathcal{O}_{\widetilde{Q}}(1))$ *does* carry a representation of G and, hence, so does $H^0(\widetilde{Q}, \mathcal{O}_{\widetilde{Q}}(k))$ for any integer k.

Note that, when acting on degree-k polynomials $p_k(z)$, the commutant eq. (4.2) becomes

$$g_1 g_2 g_1^{-1} g_2^{-1}\Big(p_k(z)\Big) = e^{2\pi i \frac{k}{5}}\, p_k(z). \tag{4.9}$$

Therefore, if and only if k is divisible by 5 then the G representation reduces to a true $\mathbb{Z}_5 \times \mathbb{Z}_5$ representation on $H^0(\widetilde{Q}, \mathcal{O}_{\widetilde{Q}}(k))$. That is, k must be of the form

$$k = 5\ell, \quad \ell \in \mathbb{Z}. \tag{4.10}$$

The formal reason for this is that only the line bundles $\mathcal{O}_{\widetilde{Q}}(5\ell)$ are $\mathbb{Z}_5 \times \mathbb{Z}_5$ equivariant. The invariant subspaces of these $\mathbb{Z}_5 \times \mathbb{Z}_5$ representations define the invariant sections. Hence, we only consider homogeneous polynomials of degrees divisible by 5 which are invariant under the action of $\mathbb{Z}_5 \times \mathbb{Z}_5$ in the following.

4.3 Invariant polynomials

As a first step, determine the $\mathbb{Z}_5 \times \mathbb{Z}_5$ invariant sections on the ambient space $\mathbb{P}^4$. That is, we must find the invariant ring

$$\mathbb{C}[z_0, z_1, z_2, z_3, z_4]^G \tag{4.11}$$

over $\mathbb{P}^4$, where G is the Heisenberg group defined in the previous subsection. One can read off the number of invariants $\hat{N}_k^G$ at each degree k from the Molien series

$$\begin{aligned} P\big(\mathbb{C}[z_0, z_1, z_2, z_3, z_4]^G, t\big) &= \sum_k \hat{N}_k^G\, t^k = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det\left(1 - tg\right)} = \\ &= 1 + 6t^5 + 41t^{10} + 156t^{15} + 426t^{20} + 951t^{25} + \\ &\quad +1856t^{30} + 3291t^{35} + 5431t^{40} + 8476t^{45} + \\ &\quad +12651t^{50} + 18206t^{55} + 25416t^{60} + 34581t^{65} + \cdots \end{aligned} \tag{4.12}$$

We see that the only invariants are of degree $k = 5\ell$, as discussed in the previous subsection. To go further than just counting the invariants, one uses the Hironaka decomposition which was introduced in section 3.3. For that, we need to choose 5 primary invariants, the same number as homogeneous coordinates. Unfortunately, any 5 out of the 6 quintic invariant polynomials are never algebraically independent. Hence, picking five degree-5 invariants never satisfies the requirements for them to be primary invariants. It turns out that the primary invariants of minimal degree consist of three degree-5 and two degree-10 invariants, which we will list in eq. (4.15) below. First, however, let us rewrite the Molien series as in eq. (3.18),

$$P\big(\mathbb{C}[z_0, z_1, z_2, z_3, z_4]^G, t\big) = \frac{1 + 3t^5 + 24t^{10} + 44t^{15} + 24t^{20} + 3t^{25} + t^{30}}{\big(1-t^5\big)^3\big(1-t^{10}\big)^2}. \tag{4.13}$$

We see that this choice of primary invariants requires

$$\frac{1}{|G|}\prod_{i=1}^{5} \deg\theta_i = \frac{5^3 10^2}{|G|} = 100 = 1 + 3 + 24 + 44 + 24 + 3 + 1 \tag{4.14}$$

secondary invariants in degrees up to 30. We again note that this decomposition is not unique, as one can always find different primary and secondary invariants. However, our choice of primary invariants is minimal, that is, leads to the least possible number (= 100) of secondary invariants.

Knowing the number of secondary invariants is not enough, however, and we need the actual G-invariant polynomials. As will be explicitly checked in section A, the five G-invariant polynomials

$$\begin{aligned}
\theta_1 &= z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 \ = z_0^5 + (\text{cyc}) \\
\theta_2 &= z_0 z_1 z_2 z_3 z_4 \\
\theta_3 &= z_0^3 z_1 z_4 + z_0 z_1^3 z_2 + z_0 z_3 z_4^3 + z_1 z_2^3 z_3 + z_2 z_3^3 z_4 \ = z_0^3 z_1 z_4 + (\text{cyc}) \\
\theta_4 &= z_0^{10} + z_1^{10} + z_2^{10} + z_3^{10} + z_4^{10} \ = z_0^{10} + (\text{cyc}) \\
\theta_5 &= z_0^8 z_2 z_3 + z_0 z_1 z_3^8 + z_0 z_2^8 z_4 + z_1^8 z_3 z_4 + z_1 z_2 z_4^8 \ = z_0^8 z_2 z_3 + (\text{cyc})
\end{aligned} \tag{4.15}$$

satisfy the necessary criterion to be our primary invariants, where (cyc) denotes the sum over the five different cyclic permutations $z_0 \to z_1 \to \cdots \to z_4 \to z_0$. Next, we need a basis for the corresponding secondary invariants, which must be of degrees 0, 5, 10, 15, 20, 25, and 30 according to eq. (4.13). In practice, these 100 secondary invariants can easily be found using Singular [41, 42]. They are

$$\eta_1 = 1, \tag{4.16a}$$

$$\eta_2 = z_0^2 z_1 z_2^2 + (\text{cyc}), \quad \eta_3 = z_0^2 z_1^2 z_3 + (\text{cyc}), \quad \eta_4 = z_0^3 z_2 z_3 + (\text{cyc}), \tag{4.16b}$$

$$\begin{aligned}
\eta_5 &= z_0^5 z_2^5 + (\text{cyc}), & \eta_6 &= z_0^4 z_2^3 z_3^3 + (\text{cyc}), & \eta_7 &= z_0^4 z_1^3 z_4^3 + (\text{cyc}),\\
\eta_8 &= z_0^4 z_1^2 z_2^4 + (\text{cyc}), & \eta_9 &= z_0^4 z_1^4 z_3^2 + (\text{cyc}), & \eta_{10} &= z_0^6 z_2^2 z_3^2 + (\text{cyc}),\\
\eta_{11} &= z_0^6 z_1^2 z_4^2 + (\text{cyc}), & \eta_{12} &= z_0^6 z_1 z_3^3 + (\text{cyc}), & \eta_{13} &= z_0^6 z_2^3 z_4 + (\text{cyc}),\\
\eta_{14} &= z_0^6 z_1^3 z_2 + (\text{cyc}), & \eta_{15} &= z_0^6 z_3 z_4^3 + (\text{cyc}), & \eta_{16} &= z_0^7 z_1 z_2^2 + (\text{cyc}),\\
\eta_{17} &= z_0^7 z_3^2 z_4 + (\text{cyc}), & \eta_{18} &= z_0^7 z_1^2 z_3 + (\text{cyc}), & \eta_{19} &= z_0^8 z_1 z_4 + (\text{cyc}),\\
\eta_{20} &= z_0^3 z_1^2 z_2^2 z_3^3 + (\text{cyc}), & \eta_{21} &= z_0^4 z_1^2 z_3^3 z_4 + (\text{cyc}), & \eta_{22} &= z_0^4 z_1 z_2^3 z_4^2 + (\text{cyc}),\\
\eta_{23} &= z_0^4 z_1^3 z_2^2 z_3 + (\text{cyc}), & \eta_{24} &= z_0^4 z_1 z_2 z_3^4 + (\text{cyc}), & \eta_{25} &= z_0^5 z_1^2 z_2 z_3^2 + (\text{cyc}),\\
\eta_{26} &= z_0^5 z_1^2 z_2^2 z_4 + (\text{cyc}), & \eta_{27} &= z_0^5 z_1 z_2^3 z_3 + (\text{cyc}), & \eta_{28} &= z_0^5 z_1^3 z_3 z_4 + (\text{cyc}),
\end{aligned} \tag{4.16c}$$

$$\begin{aligned}
\eta_{29} &= z_0^{15} + (\text{cyc}), & \eta_{30} &= z_0^{10} z_2^5 + (\text{cyc}), & \eta_{31} &= z_0^{10} z_3^5 + (\text{cyc}),\\
\eta_{32} &= z_0^{10} z_1^5 + (\text{cyc}), & \eta_{33} &= z_0^6 z_1^3 z_2^6 + (\text{cyc}), & \eta_{34} &= z_0^6 z_1^6 z_3^3 + (\text{cyc}),\\
\eta_{35} &= z_0^7 z_2^4 z_3^4 + (\text{cyc}), & \eta_{36} &= z_0^7 z_1^4 z_4^4 + (\text{cyc}), & \eta_{37} &= z_0^7 z_1^2 z_3^6 + (\text{cyc}),\\
\eta_{38} &= z_0^7 z_2^6 z_4^2 + (\text{cyc}), & \eta_{39} &= z_0^7 z_1^6 z_2^2 + (\text{cyc}), & \eta_{40} &= z_0^8 z_1^3 z_3^4 + (\text{cyc}),\\
\eta_{41} &= z_0^8 z_2^4 z_4^3 + (\text{cyc}), & \eta_{42} &= z_0^8 z_1^4 z_2^3 + (\text{cyc}), & \eta_{43} &= z_0^7 z_1^7 z_3 + (\text{cyc}),\\
\eta_{44} &= z_0^8 z_2^6 z_3 + (\text{cyc}), & \eta_{45} &= z_0^8 z_2 z_3^6 + (\text{cyc}), & \eta_{46} &= z_0^8 z_1^6 z_4 + (\text{cyc}),\\
\eta_{47} &= z_0^9 z_1^2 z_2^4 + (\text{cyc}), & \eta_{48} &= z_0^9 z_1^4 z_3^2 + (\text{cyc}), & \eta_{49} &= z_0^1 1 z_1^2 z_4^2 + (\text{cyc}),\\
\eta_{50} &= z_0^1 1 z_1 z_3^3 + (\text{cyc}), & \eta_{51} &= z_0^1 1 z_1^3 z_2 + (\text{cyc}), & \eta_{52} &= z_0^1 1 z_3 z_4^3 + (\text{cyc}),\\
\eta_{53} &= z_0^1 2 z_1 z_2^2 + (\text{cyc}), & \eta_{54} &= z_0^1 2 z_1^2 z_3 + (\text{cyc}), & \eta_{55} &= z_0^5 z_1^3 z_2^4 z_3^3 + (\text{cyc}),\\
\eta_{56} &= z_0^5 z_1^3 z_2^3 z_4^4 + (\text{cyc}), & \eta_{57} &= z_0^5 z_1^4 z_2^2 z_3^4 + (\text{cyc}), & \eta_{58} &= z_0^5 z_1^2 z_3^4 z_4^4 + (\text{cyc}),\\
\eta_{59} &= z_0^6 z_1^2 z_2^3 z_3^4 + (\text{cyc}), & \eta_{60} &= z_0^6 z_2^4 z_3^3 z_4^2 + (\text{cyc}), & \eta_{61} &= z_0^6 z_1^4 z_2^2 z_4^3 + (\text{cyc}),\\
\eta_{62} &= z_0^6 z_1^4 z_3^4 z_4 + (\text{cyc}), & \eta_{63} &= z_0^6 z_1^4 z_2^4 z_3 + (\text{cyc}), & \eta_{64} &= z_0^6 z_1 z_2^5 z_3^3 + (\text{cyc}),\\
\eta_{65} &= z_0^7 z_1^3 z_2^3 z_3^2 + (\text{cyc}), & \eta_{66} &= z_0^7 z_1^4 z_2 z_3^3 + (\text{cyc}), & \eta_{67} &= z_0^7 z_2^3 z_3 z_4^4 + (\text{cyc}),\\
\eta_{68} &= z_0^7 z_1^3 z_2^4 z_4 + (\text{cyc}), & \eta_{69} &= z_0^7 z_1 z_2^2 z_3^5 + (\text{cyc}), & \eta_{70} &= z_0^8 z_1^2 z_2^2 z_3^3 + (\text{cyc}),\\
\eta_{71} &= z_0^8 z_1 z_3^5 z_4 + (\text{cyc}), & \eta_{72} &= z_0^9 z_1 z_2 z_3^4 + (\text{cyc}), & &
\end{aligned} \tag{4.16d}$$

$$\begin{aligned}
\eta_{73} &= z_0^{20} + (\text{cyc}), & \eta_{74} &= z_0^{10} z_2^{10} + (\text{cyc}), & \eta_{75} &= z_0^{15} z_2^5 + (\text{cyc}),\\
\eta_{76} &= z_0^{15} z_1^5 + (\text{cyc}), & \eta_{77} &= z_0^7 z_1^7 z_3^6 + (\text{cyc}), & \eta_{78} &= z_0^7 z_1^6 z_2^7 + (\text{cyc}),\\
\eta_{79} &= z_0^8 z_2^6 z_3^6 + (\text{cyc}), & \eta_{80} &= z_0^8 z_1^6 z_4^6 + (\text{cyc}), & \eta_{81} &= z_0^8 z_1^4 z_2^8 + (\text{cyc}),\\
\eta_{82} &= z_0^8 z_1^8 z_3^4 + (\text{cyc}), & \eta_{83} &= z_0^9 z_1^4 z_3^7 + (\text{cyc}), & \eta_{84} &= z_0^9 z_2^7 z_4^4 + (\text{cyc}),\\
\eta_{85} &= z_0^9 z_1^7 z_2^4 + (\text{cyc}), & \eta_{86} &= z_0^9 z_3^4 z_4^7 + (\text{cyc}), & \eta_{87} &= z_0^9 z_2^8 z_3^3 + (\text{cyc}),\\
\eta_{88} &= z_0^9 z_2^3 z_3^8 + (\text{cyc}), & \eta_{89} &= z_0^9 z_1^3 z_4^8 + (\text{cyc}), & \eta_{90} &= z_0^9 z_1^2 z_2^9 + (\text{cyc}),\\
\eta_{91} &= z_0^{11} z_1^3 z_2^6 + (\text{cyc}), & \eta_{92} &= z_0^{11} z_1^6 z_3^3 + (\text{cyc}), & \eta_{93} &= z_0^{11} z_2^3 z_4^6 + (\text{cyc}),\\
\eta_{94} &= z_0^{11} z_2^7 z_3^2 + (\text{cyc}), & \eta_{95} &= z_0^{11} z_2^2 z_3^7 + (\text{cyc}), & \eta_{96} &= z_0^{11} z_1^7 z_4^2 + (\text{cyc}),
\end{aligned} \tag{4.16e}$$

$$\eta_{97} = z_0^9 z_2^8 z_3^8 + (\text{cyc}), \quad \eta_{98} = z_0^9 z_1^8 z_4^8 + (\text{cyc}), \quad \eta_{99} = z_0^9 z_1^9 z_3^7 + (\text{cyc}), \tag{4.16f}$$

$$\eta_{100} = z_0^{30} + (\text{cyc}) = z_0^{30} + z_1^{30} + z_2^{30} + z_3^{30} + z_4^{30}. \tag{4.16g}$$

The Hironaka decomposition of the ring of G-invariant homogeneous polynomials is then

$$\mathbb{C}[z_0, z_1, z_2, z_3, z_4]^G = \bigoplus_{i=1}^{100} \eta_i \, \mathbb{C}[\theta_1, \theta_2, \theta_3, \theta_4, \theta_5]. \tag{4.17}$$

As a simple application, we can read off a basis for the invariant degree-5 polynomials,

$$\begin{aligned} \mathbb{C}\big[z_0, z_1, z_2, z_3, z_4\big]_5^G &= \operatorname{span}\Big\{\eta_1\theta_1,\ \eta_1\theta_2,\ \eta_1\theta_3,\ \eta_2,\ \eta_3,\ \eta_4\Big\} \\ &= \operatorname{span}\Big\{\theta_1,\ \theta_2,\ \theta_3,\ \eta_2,\ \eta_3,\ \eta_4\Big\}. \end{aligned} \tag{4.18}$$

Note that this is the basis of invariant quintic polynomials used in eq. (4.4) to define $\widetilde{Q}(z)$.

4.4 Invariant sections on the quintic

The next step is to restrict the G-invariant sections on $\mathbb{P}^4$ to the hypersurface $\widetilde{Q}$. In section 2, we showed how to accomplish this for all sections on generic quintics $\widetilde{Q} \in \mathbb{P}^4$. Since the sections on the ambient space are nothing but homogeneous polynomials, the restricted sections were the quotient of the homogeneous polynomials by the hypersurface equation $\widetilde{Q} = 0$,

$$\begin{tikzcd} H^0\big(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(k)\big) \arrow[r, "\text{restrict}"] \arrow[d, equal] & H^0\big(\widetilde{Q}, \mathcal{O}_{\widetilde{Q}}(k)\big) \arrow[d, equal] \\ \mathbb{C}[z_0, z_1, z_2, z_3, z_4]_k \arrow[r, "\widetilde{Q}=0"] & \Big(\mathbb{C}[z_0, z_1, z_2, z_3, z_4]\big/\langle\widetilde{Q}\rangle\Big)_k . \end{tikzcd} \tag{4.19}$$

Now consider the quintics defined by eq. (4.4), which allow a $\mathbb{Z}_5 \times \mathbb{Z}_5$ action. Here, one only wants to know the G-invariant sections on $\widetilde{Q}$, since these correspond to the sections on the $\mathbb{Z}_5 \times \mathbb{Z}_5$ quotient $Q = \widetilde{Q}/(\mathbb{Z}_5 \times \mathbb{Z}_5)$. Moreover, since the G-invariant polynomials are of degree 5ℓ, we only consider this case. Hence, the G-invariant sections are

$$\begin{tikzcd} H^0\big(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5\ell)\big)^G \arrow[r, "\text{restrict}"] \arrow[d, equal] & H^0\big(\widetilde{Q}, \mathcal{O}_{\widetilde{Q}}(5\ell)\big)^G \arrow[d, equal] \\ \mathbb{C}[z_0, z_1, z_2, z_3, z_4]_{5\ell}^G \arrow[r, "\widetilde{Q}=0"] & \Big(\mathbb{C}[z_0, z_1, z_2, z_3, z_4]\big/\langle\widetilde{Q}\rangle\Big)_{5\ell}^G . \end{tikzcd} \tag{4.20}$$

Finally, we identify the invariant sections on $\widetilde{Q}$ with sections on the quotient manifold Q, as discussed in section 3.1. Therefore, the sections on Q are

$$H^0\Big(Q, \mathcal{O}_{\widetilde{Q}}(5\ell)/(\mathbb{Z}_5 \times \mathbb{Z}_5)\Big) = H^0\big(\widetilde{Q}, \mathcal{O}_{\widetilde{Q}}(5\ell)\big)^G = \Big(\mathbb{C}[z_0, z_1, z_2, z_3, z_4]\big/\langle\widetilde{Q}\rangle\Big)_{5\ell}^G . \tag{4.21}$$

By unravelling the definitions and using eq. (4.17), the invariant subspace of the quotient ring is given by

$$\begin{aligned} \Big(\mathbb{C}[z_0, z_1, z_2, z_3, z_4]\big/\langle\widetilde{Q}(z)\rangle\Big)^G &= \mathbb{C}[z_0, z_1, z_2, z_3, z_4]^G\big/\langle\widetilde{Q}(z)\rangle^G \\ &= \left(\bigoplus_{i=1}^{100} \eta_i \, \mathbb{C}\big[\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\big]\right) \Bigg/ \left(\bigoplus_{i=1}^{100} \widetilde{Q}\, \eta_i \, \mathbb{C}\big[\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\big]\right). \end{aligned} \tag{4.22}$$

5ℓ	5	10	15	20	25	30	35	40
$\hat{N}^G_{5\ell}$	6	41	156	426	951	1856	3291	5431
$N^G_{5\ell}$	5	35	115	270	525	905	1435	2140

Table 2: The number of G-invariant degree 5ℓ-homogeneous polynomials $\hat{N}^G_{5\ell}$, eq. (4.12), and the number of remaining invariant polynomials $N^G_{5\ell}$ after imposing the hypersurface equation $\widetilde{Q}(z) = 0$, see eq. (4.4).

Using eq. (4.4), the hypersurface equation is

$$\begin{aligned} \widetilde{Q}(z) &= 0 & \Leftrightarrow \\ z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 &= -\phi_1\big(z_0 z_1 z_2 z_3 z_4\big) - \cdots & \Leftrightarrow \\ \theta_1 &= -\phi_1\theta_2 - \phi_2\theta_3 - \phi_3\eta_2 - \phi_4\eta_3 - \phi_5\eta_4, \end{aligned} \tag{4.23}$$

and, hence, we can simply eliminate θ_1. Therefore, forming the quotient is particularly easy, and we obtain

$$\Big(\mathbb{C}[z_0, z_1, z_2, z_3, z_4]\Big/\big\langle\widetilde{Q}(z)\big\rangle\Big)^G = \bigoplus_{i=1}^{100} \eta_i\, \mathbb{C}[\theta_2, \theta_3, \theta_4, \theta_5]. \tag{4.24}$$

We list the number $\hat{N}^G_{5\ell}$ of G-invariant degree-5ℓ polynomials on $\mathbb{P}^4$ as well as the number of invariant polynomials after restricting to $\widetilde{Q}$, $N^G_{5\ell}$, in table 2. Since we know the homogeneous degrees of the primary and secondary invariants, θ and η respectively, it is a simple combinatorial problem to list all $N^G_{5\ell}$ monomials in eq. (4.24) of fixed degree 5ℓ. They then form a basis for the sections on Q,

$$\begin{aligned} H^0\Big(Q, \mathcal{O}_{\widetilde{Q}}(5\ell)/(\mathbb{Z}_5\times\mathbb{Z}_5)\Big) &= \operatorname{span}\big\{s_\alpha\big\}_{\alpha=0}^{N^G_{5\ell}-1} \\ &= \Bigg(\bigoplus_{i=1}^{100} \eta_i\, \mathbb{C}[\theta_2, \theta_3, \theta_4, \theta_5]\Bigg)_{5\ell} = \bigoplus_{i=1}^{100} \eta_i\, \mathbb{C}[\theta_2, \theta_3, \theta_4, \theta_5]_{5\ell - \deg \eta_i}. \end{aligned} \tag{4.25}$$

4.5 Results

We have now computed an explicit basis of invariant sections of $\mathcal{O}_{\widetilde{Q}}(5\ell)$, which can be identified with a basis of sections on the quotient manifold $Q = \widetilde{Q}\big/(\mathbb{Z}_5\times\mathbb{Z}_5)$. This is all we need to extend Donaldson's algorithm to Q. Literally the only difference in the computer program used in section 2.4 is that now

- the degree of the polynomials must be $k = 5\ell$, $\ell \in \mathbb{Z}_>$, and
- the sections are given in eq. (4.25).

Hence, one can compute the balanced metrics on Q. As $\ell \to \infty$, these will approach the unique Calabi-Yau metric. We write $\sigma_{5\ell}(Q)$ for the error measure computed directly for the balanced metrics on the non-simply connected threefold Q. Note that there is still a 5-dimensional complex structure moduli space of such threefolds. However, as we have

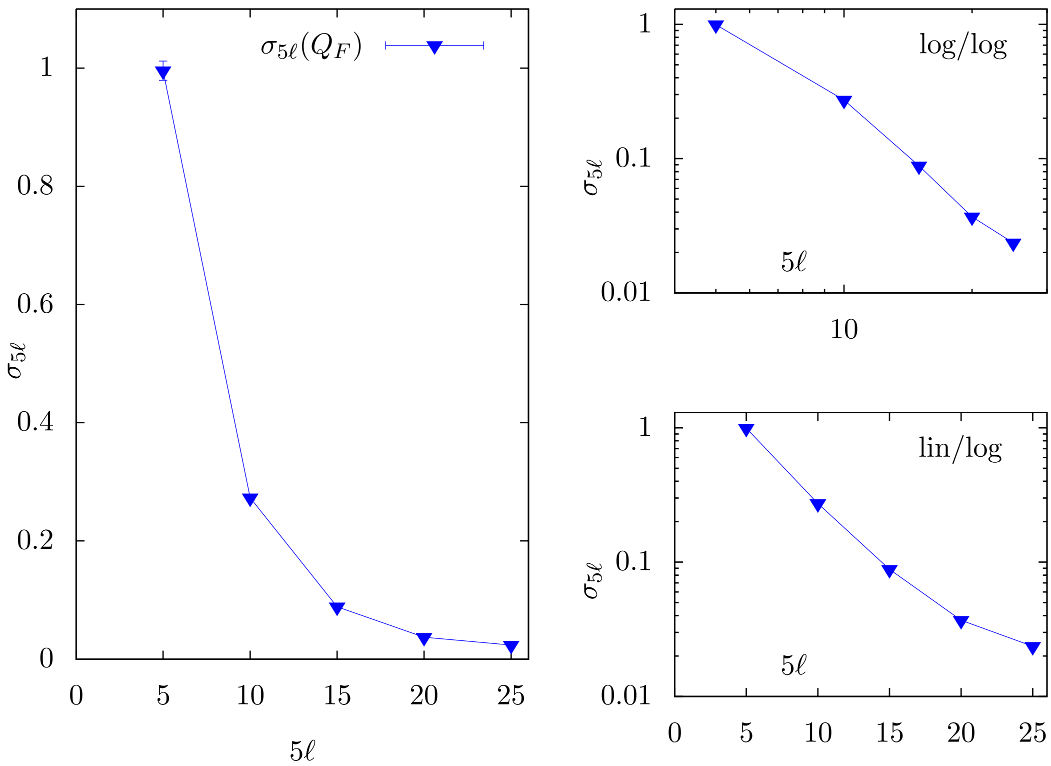

Figure 5: The error measure $\sigma_{5\ell}(Q_F)$ on the non-simply connected threefold $Q_F = \widetilde{Q}_F/(\mathbb{Z}_5 \times \mathbb{Z}_5)$. For each $\ell \in \mathbb{Z}_>$ we iterated the T-operator 10 times, numerically integrating using $N_p = 1{,}000{,}000$ points. Then we evaluated $\sigma_{5\ell}(Q_F)$ using 20,000 different test points. Note that all three plots show the same data, but with different combinations of linear and logarithmic axes.

seen in figure 4, the details of the complex structure essentially play no role in how fast the balanced metrics converge to the Calabi-Yau metric. Therefore, as an example, in figure 5 we plot $\sigma_{5\ell}$ for the quotient $Q_F = \widetilde{Q}_F/(\mathbb{Z}_5 \times \mathbb{Z}_5)$ of the Fermat quintic. Note that the error measure tends to zero as $\ell \to \infty$, as it should.

Comparison with the covering space. We have now extended Donaldson's algorithm so as to compute the successive approximations to the Calabi-Yau metric directly on the quotient manifold Q. Clearly, these metrics can be pulled back to $\mathbb{Z}_5 \times \mathbb{Z}_5$ symmetric metrics on the covering space $\widetilde{Q}$, thus approximating the Calabi-Yau metric on $\widetilde{Q}$. Let us denote by $\omega_{5\ell}$ the balanced Kähler form on Q computed at degree-5ℓ, and by $q^*\omega_{5\ell}$ its pull-back to $\widetilde{Q}$. We define $\tilde{\sigma}_{5\ell}(\widetilde{Q})$ to be the error measure evaluated using the pull-back

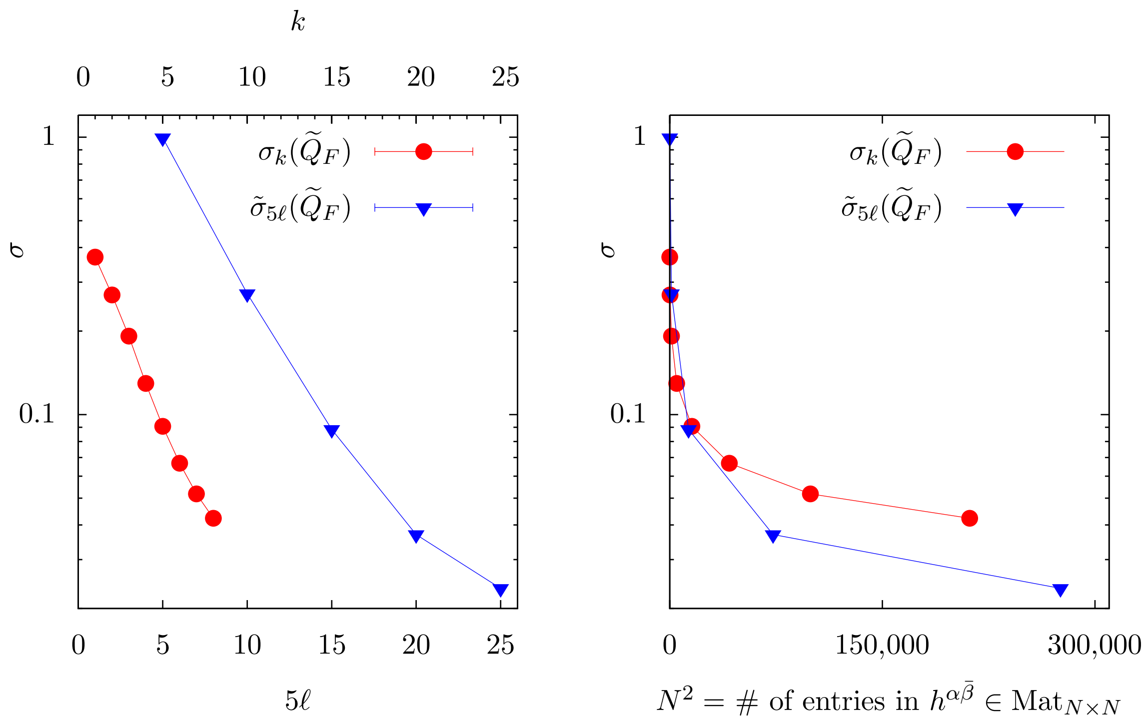

Figure 6: The metric pulled back from $Q_F = \widetilde{Q}_F/(\mathbb{Z}_5\times\mathbb{Z}_5)$ compared with the metric computation on $\widetilde{Q}_F$. The error measures are $\tilde{\sigma}_{5\ell}(\widetilde{Q}_F)$ and $\sigma_k(\widetilde{Q}_F)$, respectively. On the left, we plot them by the degree of the homogeneous polynomials. On the right, we plot them as a function of N^2, the number of sections squared. On Q_F, the number of sections is $N^G_{5\ell}$; on $\widetilde{Q}_F$ the number of sections is N_k. The σ-axis is logarithmic.

metric, that is,

$$\begin{aligned}\tilde{\sigma}_{5\ell}(\widetilde{Q}) &= \frac{1}{\mathrm{Vol}_{\mathrm{CY}}\,(\widetilde{Q})}\int_{\widetilde{Q}}\left|1-\frac{q^*\omega_{5\ell}^3\Big/\,\mathrm{Vol}_{\mathrm{K}}\,(\widetilde{Q})}{\Omega\wedge\bar{\Omega}\Big/\,\mathrm{Vol}_{\mathrm{CY}}\,(\widetilde{Q})}\right|\mathrm{dVol}_{\mathrm{CY}} = \\ &= \frac{1}{\mathrm{Vol}_{\mathrm{CY}}(Q)}\int_{Q}\left|1-\frac{\omega_{5\ell}^3\Big/\,\mathrm{Vol}_{\mathrm{K}}(Q)}{\Omega\wedge\bar{\Omega}\Big/\,\mathrm{Vol}_{\mathrm{CY}}(Q)}\right|\mathrm{dVol}_{\mathrm{CY}} = \sigma_{5\ell}(Q). \qquad (4.26)\end{aligned}$$

Now recall that in section 2 it was shown how to determine the Calabi-Yau metric on *any* quintic threefold. This, of course, includes the $\mathbb{Z}_5\times\mathbb{Z}_5$ quintics $\widetilde{Q}$ defined by eq. (4.4). However, since most quintics do not admit a finite group action, the procedure specified in section 2 finds the Calabi-Yau metric using *generic* homogeneous polynomials. That is, it finds an explicit polynomial basis for $H^0(\widetilde{Q},\mathcal{O}_{\widetilde{Q}}(k))$, computes the balanced metric and determines the Calabi-Yau metric as the $k\to\infty$ limit. When applied to our $\mathbb{Z}_5\times\mathbb{Z}_5$ quintics, this second method will also compute the unique $\mathbb{Z}_5\times\mathbb{Z}_5$ symmetric Calabi-Yau metric. However, it does so as the limit of balanced metrics constructed from sections of $\mathcal{O}_{\widetilde{Q}}(k)$ which do not share this symmetry, rather than from invariant sections of $\mathcal{O}_{\widetilde{Q}}(5\ell)$ as above. That is, this second method does *not* exploit the $\mathbb{Z}_5\times\mathbb{Z}_5$ symmetry. The associated error measure σ_k is evaluated using eq. (2.50) for $\mathbb{Z}_5 \times \mathbb{Z}_5$-symmetric quintics. It is of some interest to compare these two methods for calculating the Calabi-Yau metric on $\widetilde{Q}$. Specifically, in the left plot of figure 6 we compare the error measure $\tilde{\sigma}_{5\ell}$ to σ_k on the Fermat quintic $\widetilde{Q}_F$. Interestingly, for fixed degrees $k = 5\ell$ the pull-back metric is a worse approximation to the Calabi-Yau metric on $\widetilde{Q}$ than the metric computed on $\widetilde{Q}$ without taking the symmetry into account. The reason is that, in addition to the $\mathbb{Z}_5 \times \mathbb{Z}_5$ invariant polynomials on $\widetilde{Q}$, there are many more that transform with some character of $\mathbb{Z}_5 \times \mathbb{Z}_5$. These polynomials provide extra degrees of freedom at fixed degree 5ℓ, which allow the balanced metric to be a better fit to the Calabi-Yau metric.

However, a more just comparison is by the amount of the numerical effort, that is, the number $(N^G_{5\ell})^2$ and $(N_k)^2$, respectively, of entries in the $h^{\alpha\bar{\beta}}$ matrix. We plot $\tilde{\sigma}_{5\ell}$ and σ_k as a function of N^2 in figure 6. We see that, except for the two lowest-degree cases $5\ell = k = 5$ and $5\ell = k = 10$, the pull-back metric computation (that is, using invariant sections) is more efficient.

5. Schoen threefolds

5.1 As complete intersections

By definition, Schoen type Calabi-Yau threefolds are the fiber product of two dP_9 surfaces, B_1 and B_2, fibered over $\mathbb{P}^1$. Recall that a dP_9 surface is defined as a blow-up of $\mathbb{P}^2$ at 9 points. In principle, these points can be "infinitesimally close", that is, one of the blow-up points lies within a previous blow-up, but we will only consider the generic case where all 9 points are distinct. Moreover, we are going to restrict ourselves to the case where "no Kodaira fibers collide". In that case, the Hodge diamond of the Schoen threefold $\widetilde{X}$ is [4, 36, 43]

$$h^{p,q}\big(\widetilde{X}\big) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 19 & & 0 & \\ 1 & & 19 & & 19 & & 1 \\ & 0 & & 19 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{5.1}$$

These generic Schoen Calabi-Yau threefolds can be written as a complete intersection as follow [12, 13, 37, 44]. First, consider the ambient variety $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ with coordinates

$$\Big([x_0 : x_1 : x_2],\ [t_0 : t_1],\ [y_0 : y_1 : y_2]\Big) \in \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2 . \tag{5.2}$$

The Calabi-Yau threefold $\widetilde{X}$ is then cut out as the zero-set of two equations of multi-degrees $(3, 1, 0)$ and $(0, 1, 3)$, respectively. The two equations are of the form

$$\widetilde{P}(x,t,y) \ = t_0 \widetilde{P}_1\big(x_0, x_1, x_2\big) + t_1 \widetilde{P}_2\big(x_0, x_1, x_2\big) = 0, \tag{5.3a}$$

$$\widetilde{R}(x,t,y) \ = t_1 \widetilde{R}_1\big(y_0, y_1, y_2\big) + t_0 \widetilde{R}_2\big(y_0, y_1, y_2\big) = 0 \tag{5.3b}$$

where $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{R}_1$, and $\widetilde{R}_2$ are cubic polynomials. The ambient space $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ is a toric variety and $\widetilde{X}$ is a toric complete intersection Calabi-Yau threefold [45 – 47].

5.2 Line bundles and sections

The first Chern classes of line bundles on $\widetilde{X}$ form a

$$h^{1,1}\left(\widetilde{X}\right) = 19 \tag{5.4}$$

dimensional lattice. Note, however, that most of them do not come from the ambient space which has

$$h^{1,1}\big(\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2\big) = 3. \tag{5.5}$$

In other words, most of the divisors D and their associated line bundles $\mathcal{L}(D)$ are not toric; that is, they cannot be described by toric methods. We could embed $\widetilde{X}$ in a much more complicated toric variety [46] where all divisors are toric. However, for now[8] we will simply ignore the non-toric divisors and restrict ourselves to line bundles on $\widetilde{X}$ that are induced from $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$.

The line bundles on $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ are classified by their first Chern class

$$c_1\Big(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^2}(a_1,b,a_2)\Big) = (a_1,b,a_2) \in \mathbb{Z}^3 = H^2\big(\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2, \mathbb{Z}\big). \tag{5.6}$$

Just as in the $\mathbb{P}^4$ case previously, their sections are homogeneous polynomials of the homogeneous coordinates. Now, however, there are three independent degrees, one for each factor. That is, the sections of $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^2}(a_1,b,a_2)$ are homogeneous polynomials of

- degree a_1 in x_0, x_1, x_2,
- degree b in t_0, t_1,
- degree a_2 in y_0, y_1, y_2.

The number of such polynomials (that is, the dimension of the linear space of polynomials) is counted by the Poincaré series

$$\begin{aligned} P\Big(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^2},(x,t,y)\Big) &= \sum_{a_1,b,a_2} h^0\Big(\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^2, \mathcal{O}(a_1,b,a_2)\Big) x^{a_1} t^b y^{a_2} \\ &= \frac{1}{(1-x)^3}\frac{1}{(1-t)^2}\frac{1}{(1-y)^3}. \end{aligned} \tag{5.7}$$

We now want to restrict the sections to the complete intersection $\widetilde{X} \subset \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$; that is, find the image

$$H^0\Big(\mathbb{P}^2\times\mathbb{P}^1\times\mathbb{P}^2, \mathcal{O}(a_1,b,a_2)\Big) \xrightarrow{\text{restrict}} H^0\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(a_1,b,a_2)\Big) \longrightarrow 0 \tag{5.8}$$

for[9] $a_1, b, a_2 > 0$. As discussed previously, this amounts to finding a basis for the quotient space

$$H^0\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(a_1,b,a_2)\big) = \Big(\mathbb{C}\big[x_0,x_1,x_2,t_0,t_1,y_0,y_1,y_2\big] \Big/ \langle \widetilde{P}, \widetilde{R}\rangle\Big)_{(a_1,b,a_2)} \tag{5.9}$$

[8]This will be partially justified in section 6, where we investigate a certain $\mathbb{Z}_3 \times \mathbb{Z}_3$-quotient of $\widetilde{X}$. There, only the toric line bundles will be relevant.

[9]Note that $c_1\big(\mathcal{O}_{\widetilde{X}}(a_1,b,a_2)\big) \in H^2(X,\mathbb{Z})$ is in the interior of the Kähler cone if and only if $a_1, b, a_2 > 0$, see [48].

of degree (a_1, b, a_2). Note that this quotient by more than one polynomial is much more difficult than the case where one quotients out a single polynomial, as we did for quintics in section 2. In general, this requires the technology of Gröbner bases [49]. Suffices to say that we are in a very advantageous position here.

By a suitable coordinate change, we can assume that the $t_0 y_0^3$ term in $\widetilde{R}$ is absent. That is,

$$\begin{aligned} \widetilde{P} &= t_0 x_0^3 + \cdots \\ \widetilde{R} &= 0 \cdot t_0 y_0^3 + t_0 y_0^2 y_1 + \cdots . \end{aligned} \tag{5.10}$$

Then, for otherwise generic polynomials $\widetilde{P}$ and $\widetilde{R}$ and lexicographic monomial order

$$x_0 \prec y_0 \prec t_0 \prec x_1 \prec y_1 \prec t_1 \prec x_2 \prec y_2, \tag{5.11}$$

the two polynomials generating

$$\langle \widetilde{P}, \widetilde{R} \rangle \subset \mathbb{C}[x_0, x_1, x_2, t_0, t_1, y_0, y_1, y_2] \tag{5.12}$$

already form a Gröbner basis. This means that the quotient in eq. (5.9) can be implemented simply by eliminating the leading monomials $t_0 x_0^3$ and $t_0 y_0^2 y_1$ in the polynomial ring $\mathbb{C}[x_0, x_1, x_2, t_0, t_1, y_0, y_1, y_2]$.

5.3 The Calabi-Yau volume form

As in the case of a hypersurface, one can express the $(3,0)$-form of the complete intersection as a Griffiths residue. By definition, the zero loci $\widetilde{P} = 0$ and $\widetilde{R} = 0$ intersect transversally, so one can encircle each in an independent transverse direction. The double residue integral

$$\Omega = \oint \oint \frac{\mathrm{d}^2 x \, \mathrm{d}t \, \mathrm{d}^2 y}{\widetilde{P} \cdot \widetilde{R}} \tag{5.13}$$

is again independent of the chosen inhomogeneous coordinate chart. Hence, it defines a holomorphic $(3,0)$-form which must be the holomorphic volume form.

5.4 Generating points

Since the defining Equations (5.3a), (5.3b) are at most cubic in the x and y coordinates, there is a particularly nice way to pick points. This is a generalization of the $L \cap \widetilde{Q}$ method presented in section 2.3 to generate points in generic quintics. In the present case, select a specific $\mathbb{P}^1 \times \mathbb{P}^1$ in the ambient space, namely,

$$\mathbb{P}^1 \times \{\text{pt.}\} \times \mathbb{P}^1 \;\subset\; \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2 . \tag{5.14}$$

This can easily be done with an $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$-invariant probability density of such configurations. The intersection

$$\Big(\mathbb{P}^1 \times \{\text{pt.}\} \times \mathbb{P}^1 \Big) \cap \widetilde{X} = \{9 \text{ points}\} \tag{5.15}$$

consists of nine points. To compute the coordinates of the nine points, one needs to solve two cubic equations, which can be done analytically.[10]

We still need the distribution of these "random" points. First, note that there are three obvious $(1,1)$-forms. These are the pull-backs

$$\pi_1^*\big(\omega_{\mathbb{P}^2}\big), \quad \pi_2^*\big(\omega_{\mathbb{P}^1}\big), \quad \pi_3^*\big(\omega_{\mathbb{P}^2}\big) \tag{5.16}$$

of the standard ($\mathrm{SU}(m+1)$ symmetric) Fubini-Study Kähler forms on $\mathbb{P}^m$, where π_i is the projection on the i-th factor of the ambient space. However, here the $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{SU}(3)$ symmetry of the ambient space is not enough to determine the distribution of points uniquely.

In general, the question about the distribution of zeros was answered by Shifman and Zelditch [50]. Let us quickly review the result. Let $\mathcal{L}$ be a line bundle on a complex manifold Y and pick a basis $s_0, \dots, s_{N-1}$ of sections

$$\operatorname{span}\big\{s_0, \dots, s_{N-1}\big\} = H^0(Y, \mathcal{L}). \tag{5.17}$$

Moreover, let $\mathcal{L}$ be base-point free, that is, the sections do not have a common zero. In other words,

$$\Phi_{\mathcal{L}} : Y \to \mathbb{P}^{N-1}, \; x \mapsto \big[s_0(t) : s_1(t) : \cdots : s_{N-1}(t)\big] \tag{5.18}$$

is a well-defined map. The sections generate the N-dimensional vector space $H^0(Y, \mathcal{L})$ which contains the unit sphere $SH^0(Y, \mathcal{L})$. In other words, if we define $s_0, \dots, s_{N-1}$ to be an orthonormal basis, then $SH^0(Y, \mathcal{L})$ is the common $\mathrm{SU}(N)$-orbit of the basis sections. We take a random section $s \in SH^0(Y, \mathcal{L})$ to be uniformly distributed with respect to the usual "round" measure, that is, $\mathrm{SU}(N)$-uniformly distributed.

Finally, switch from each such section s to its zero locus Z_s in Y, and consider the expected distribution of the random zero loci. Then

Theorem 5 (Shifman, Zelditch) *Under the above assumptions (in particular, that $\Phi_{\mathcal{L}}$ is well-defined) the expected distribution of zero loci Z_s is*

$$\big\langle Z_{\mathcal{L}} \big\rangle = \frac{1}{N} \Phi_{\mathcal{L}}^* \omega_{FS}, \tag{5.19}$$

where ω_{FS} is the standard Fubini-Study Kähler form on $\mathbb{P}^{N-1}$.

Note that, in our case, the embedding $\widetilde{X} \subset \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ is generated by the three line bundles

$$\begin{aligned}
H^0\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1,0,0)\big) &= \operatorname{span}\{x_0, x_1, x_2\} &&\Rightarrow && \Phi_{\mathcal{O}_{\widetilde{X}}(1,0,0)} : && \widetilde{X} \to \mathbb{P}^2, \\
H^0\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(0,1,0)\big) &= \operatorname{span}\{t_0, t_1\} &&\Rightarrow && \Phi_{\mathcal{O}_{\widetilde{X}}(0,1,0)} : && \widetilde{X} \to \mathbb{P}^1, \\
H^0\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(0,0,1)\big) &= \operatorname{span}\{y_0, y_1, y_2\} &&\Rightarrow && \Phi_{\mathcal{O}_{\widetilde{X}}(0,0,1)} : && \widetilde{X} \to \mathbb{P}^2.
\end{aligned} \tag{5.20}$$

[10]Recall that, to generate points on the quintic, we had to solve a quintic polynomial. This can only be done numerically.

(a_1, b, a_2)	$(1,1,1)$	$(2,2,2)$	$(3,3,3)$	$(4,4,4)$	$(5,5,5)$	$(6,6,6)$
$\hat{N}_{(a_1,b,a_2)}$	18	108	400	1125	2646	5488
$N_{(a_1,b,a_2)}$	18	108	343	801	1566	2728

Table 3: The number of degree (a_1, b, a_2)-homogeneous polynomials $\hat{N}_{(a_1,b,a_2)}$ over $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ and the number of remaining polynomials $N_{(a_1,b,a_2)}$ on $\widetilde{X}$ after imposing the two equalities $\widetilde{P} = 0 = \widetilde{R}$ defining the complete intersection.

Although none of the three Φ maps is an embedding, they are all well-defined. This is sufficient for the theorem of Shifman and Zelditch. We point out that the Φ maps are nothing but the restriction of the projections π to $\widetilde{X} \subset \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$,

$$\Phi_{\mathcal{O}_{\widetilde{X}}(1,0,0)} = \pi_1\big|_{\widetilde{X}}, \quad \Phi_{\mathcal{O}_{\widetilde{X}}(0,1,0)} = \pi_2\big|_{\widetilde{X}}, \quad \Phi_{\mathcal{O}_{\widetilde{X}}(0,0,1)} = \pi_3\big|_{\widetilde{X}}. \tag{5.21}$$

Hence, the expected distribution of a zero-loci of sections on $\widetilde{X}$ is

$$\Big\langle Z_{\mathcal{O}_{\widetilde{X}}(1,0,0)}\Big\rangle \sim \pi_1^*\big(\omega_{\mathbb{P}^2}\big)\big|_{\widetilde{X}}, \quad \Big\langle Z_{\mathcal{O}_{\widetilde{X}}(0,1,0)}\Big\rangle \sim \pi_2^*\big(\omega_{\mathbb{P}^1}\big)\big|_{\widetilde{X}}, \quad \Big\langle Z_{\mathcal{O}_{\widetilde{X}}(0,0,1)}\Big\rangle \sim \pi_3^*\big(\omega_{\mathbb{P}^2}\big)\big|_{\widetilde{X}}. \tag{5.22}$$

These are precisely the three $(1,1)$-forms we introduced previously in eq. (5.16). Therefore, if we independently pick the two $\mathbb{P}^1$ factors and the point in eq. (5.14), then the distribution of simultaneous zero loci is

$$\mathrm{d}A \sim \pi_1^*\big(\omega_{\mathbb{P}^2}\big) \wedge \pi_2^*\big(\omega_{\mathbb{P}^1}\big) \wedge \pi_3^*\big(\omega_{\mathbb{P}^2}\big)\Big|_{\widetilde{X}}. \tag{5.23}$$

In other words, the points generated by the above algorithm are randomly distributed with respect to the auxiliary measure $\mathrm{d}A$.

5.5 Results

The new feature of the Schoen Calabi-Yau threefold, as opposed to the quintic, is that one now has different directions in the Kähler moduli space. On quintic threefolds there is only one Kähler modulus, which is just the overall volume. Now, however, there is a $19 = h^{1,1}(\widetilde{X})$ dimensional Kähler moduli space of which we parametrize 3 directions by the toric line bundles $\mathcal{O}_{\widetilde{X}}(a_1, b, a_2)$. Note that, here as elsewhere in algebraic geometry, one has to work with integral Kähler classes that are the first Chern classes of some line bundle. This is not a real restriction, however, since any irrational slope direction in the Kähler moduli space can be approximated by a rational slope. A line with rational slope always intersects points in $H^2(\widetilde{X}, \mathbb{Z})$.

By way of an example, choose the direction $(1,1,1)\mathbb{Z}_> \subset H^2(\widetilde{X}, \mathbb{Z})$ in the Kähler moduli space; that is, the line bundles of the form $\mathcal{O}_{\widetilde{X}}(k,k,k)$ for $k \in \mathbb{Z}$, $k > 0$. We list in table 3 the number of sections in both $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ and in its restriction to the Schoen manifold $\widetilde{X}$. Note that they grow very fast with k, and quickly grow outside of the range amenable to computation. However, the degree of accuracy of the metric on $\widetilde{X}$ is essentially determined by $N^2_{(k,k,k)}$, the number of metric parameters that we fit to approximate the Calabi-Yau metric. Recall from the Hodge diamond eq.(5.1) that the complex structure moduli space is $19 = h^{2,1}(\widetilde{X})$-dimensional. However, as in figure 4, the convergence of

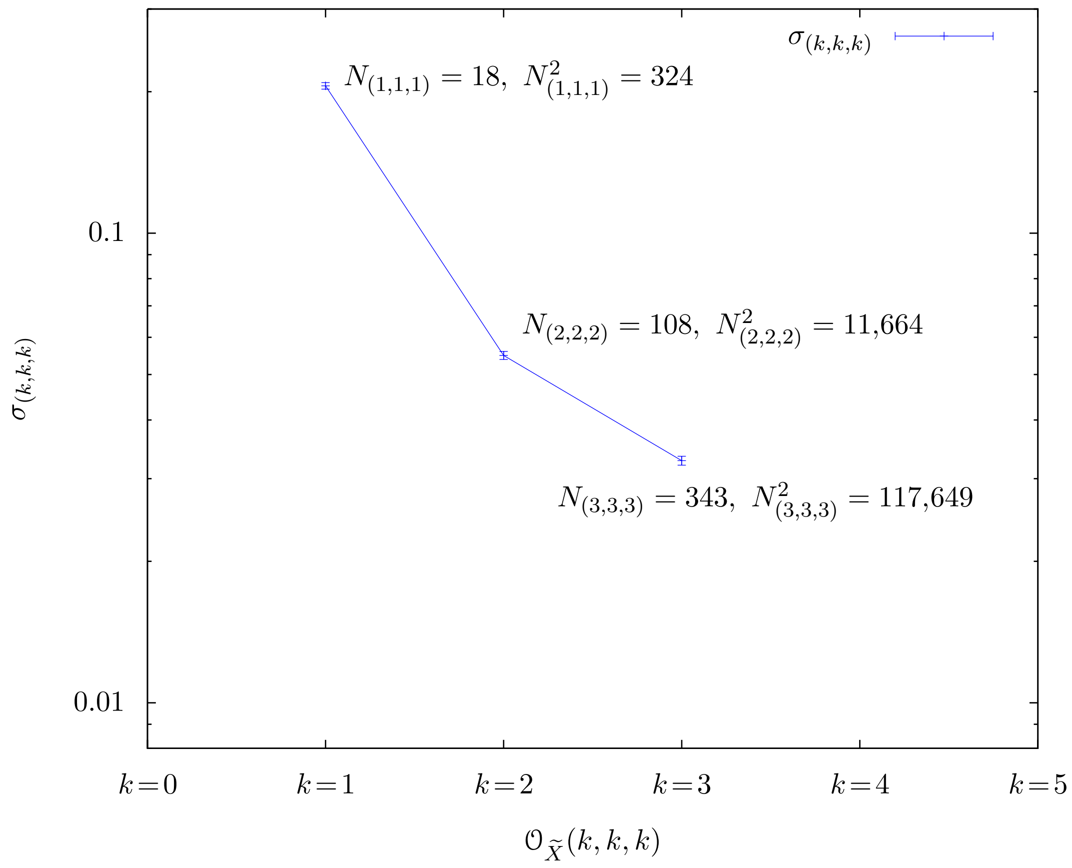

Figure 7: The error measure $\sigma_{(k,k,k)}$ for the metric on a $\mathbb{Z}_3 \times \mathbb{Z}_3$ Schoen threefold $\widetilde{X}$. We iterated the T-operator 5 times, numerically integrating using $N_p = 1{,}000{,}000$ points. Finally, we integrated $\sigma_{(k,k,k)}$ using 10,000 points. $N_{(k,k,k)}$ is the number of sections $h^0\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(k,k,k)\big)$.

the balanced metrics is essentially independent of the choice of complex structure. Hence, as an example, we choose a specific $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetric Schoen threefold ($\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1$) defined in the next section. In figure 7, we plot the error measure $\sigma_{(k,k,k)}$ vs. k for this manifold and find very fast convergence. Note how the $k = 3$ data point already approaches to within 10% of the limit $N_p(= 10^6) > N^2_{(k,k,k)}(= 117,649)$, but still yields a quite small value of $\sigma_{(3,3,3)} \approx 4 \times 10^{-2}$.

6. The $\mathbb{Z}_3 \times \mathbb{Z}_3$ manifold

6.1 A symmetric Schoen threefold

For special complex structures, the Schoen Calabi-Yau threefold has a free $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action [44, 51], which we now describe. Recall that the Schoen threefolds can be written as complete intersections in

$$\Big([x_0 : x_1 : x_2],\ [t_0 : t_1],\ [y_0 : y_1 : y_2]\Big) \in \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2, \tag{6.1}$$

as discussed in section 5. Let us start by defining the $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action on the ambient space [36], where it is generated by ($\omega = e^{\frac{2\pi i}{3}}$)

$$\gamma_1 : \begin{cases} [x_0 : x_1 : x_2] \mapsto [x_0 : \omega x_1 : \omega^2 x_2] \\ [t_0 : t_1] \mapsto [t_0 : \omega t_1] \\ [y_0 : y_1 : y_2] \mapsto [y_0 : \omega y_1 : \omega^2 y_2] \end{cases} \tag{6.2a}$$

and

$$\gamma_2 : \begin{cases} [x_0 : x_1 : x_2] \mapsto [x_1 : x_2 : x_0] \\ [t_0 : t_1] \mapsto [t_0 : t_1] \text{ (no action)} \\ [y_0 : y_1 : y_2] \mapsto [y_1 : y_2 : y_0]. \end{cases} \tag{6.2b}$$

The two generators commute up to phases on each of the two $\mathbb{P}^2$ factors and, hence, define a $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action on the ambient space. Note that γ_2 acts non-torically, that is, not by a phase rotation. In order to define a $\mathbb{Z}_3 \times \mathbb{Z}_3$-symmetric Calabi-Yau threefold, we have to ensure that the zero locus $\widetilde{P} = 0 = \widetilde{R}$ is mapped to itself by the group action. For that to be the case, one must restrict the polynomials $\widetilde{P}$ and $\widetilde{R}$ to have a special form. It was shown in [36] that one need only constrain the cubic polynomials $\widetilde{P}_1$, $\widetilde{P}_2$, $\widetilde{R}_1$, $\widetilde{R}_2$ in eqs. (5.3a) and (5.3b). Specifically, the $\mathbb{Z}_3 \times \mathbb{Z}_3$-symmetric Schoen Calabi-Yau threefolds are defined by the simultaneous vanishing of the two polynomials

$$\begin{aligned} \widetilde{P}(x,t,y) &= t_0 \widetilde{P}_1\big(x_0,x_1,x_2\big) + t_1 \widetilde{P}_2\big(x_0,x_1,x_2\big) \\ \widetilde{R}(x,t,y) &= t_1 \widetilde{R}_1\big(y_0,y_1,y_2\big) + t_0 \widetilde{R}_2\big(y_0,y_1,y_2\big), \end{aligned} \tag{6.3}$$

where

$$\begin{aligned} \widetilde{P}_1\big(x_0,x_1,x_2\big) &= x_0^3 + x_1^3 + x_2^3 + \lambda_1 x_0 x_1 x_2 \\ \widetilde{P}_2\big(x_0,x_1,x_2\big) &= \lambda_3\big(x_0^2 x_2 + x_1^2 x_0 + x_2^2 x_1\big) \\ \widetilde{R}_1\big(y_0,y_1,y_2\big) &= y_0^3 + y_1^3 + y_2^3 + \lambda_2 y_0 y_1 y_2 \\ \widetilde{R}_2\big(y_0,y_1,y_2\big) &= y_0^2 y_1 + y_1^2 y_2 + y_2^2 y_0. \end{aligned} \tag{6.4}$$

In the following, we will always take $\widetilde{P}$, $\widetilde{R}$ to be of this form. Note that, up to coordinate changes, the polynomials depend on 3 complex parameters λ_1, λ_2, and λ_3.

One can easily check that $\widetilde{P}$ is completely invariant under the $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action, as one naively expects. However, $\widetilde{R}$ is not quite invariant. Rather, it transforms like a character of $\mathbb{Z}_3 \times \mathbb{Z}_3$. That is,

$$\widetilde{P}(\gamma_1 x, \gamma_1 t, \gamma_1 y) = \widetilde{P}(x,t,y) \qquad \widetilde{P}(\gamma_2 x, \gamma_2 t, \gamma_2 y) = \widetilde{P}(x,t,y) \tag{6.5}$$

$$\widetilde{R}(\gamma_1 x, \gamma_1 t, \gamma_1 y) = e^{\frac{2\pi i}{3}} \widetilde{R}(x,t,y) \qquad \widetilde{R}(\gamma_2 x, \gamma_2 t, \gamma_2 y) = \widetilde{R}(x,t,y). \tag{6.6}$$

Nevertheless, the zero set $\widetilde{P} = 0 = \widetilde{R}$ *is* invariant under the group action. Moreover, the fixed point sets of γ_1 and γ_2 on the ambient space $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ are

$$\begin{gathered} \Big\{[1{:}0{:}0],\ [0{:}1{:}0],\ [0{:}0{:}1]\Big\} \times \Big\{[0{:}1],\ [1{:}0]\Big\} \times \Big\{[1{:}0{:}0],\ [0{:}1{:}0],\ [0{:}0{:}1]\Big\}, \\ \Big\{[1{:}1{:}1],\ [1{:}\omega{:}\omega^2],\ [1{:}\omega^2{:}\omega]\Big\} \times \mathbb{P}^1 \times \Big\{[1{:}1{:}1],\ [1{:}\omega{:}\omega^2],\ [1{:}\omega^2{:}\omega]\Big\}, \end{gathered} \tag{6.7}$$

respectively. For generic[11] λ_i, the Calabi-Yau threefold $\widetilde{X}$ misses the $\mathbb{Z}_3 \times \mathbb{Z}_3$-fixed points. Therefore, the quotient

$$X = \widetilde{X} \Big/ \big(\mathbb{Z}_3 \times \mathbb{Z}_3\big) = \Big\{ \widetilde{P} = 0 = \widetilde{R} \Big\} \Big/ \big(\mathbb{Z}_3 \times \mathbb{Z}_3\big) \tag{6.8}$$

is a smooth Calabi-Yau threefold with fundamental group $\pi_1(X) = \mathbb{Z}_3 \times \mathbb{Z}_3$. Its Hodge diamond is given by [36]

$$h^{p,q}\big(X\big) = h^{p,q}\Big(\widetilde{X} \Big/ \big(\mathbb{Z}_3 \times \mathbb{Z}_3\big)\Big) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 3 & & 0 & \\ 1 & & 3 & & 3 & & 1 \\ & 0 & & 3 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} . \tag{6.9}$$

The complex structure moduli space is $h^{2,1}(X) = 3$-dimensional and parametrized by λ_1, λ_2, and λ_3.

6.2 Invariant polynomials

As discussed in section 5.2, sections of line bundles on $\widetilde{X}$ are homogeneous polynomials in $[x_0 : x_1 : x_2]$, $[t_0 : t_1]$ and $[y_0 : y_1 : y_2]$, modulo the ideal $\langle \widetilde{P}, \widetilde{R} \rangle$. We now want to consider the quotient $X = \widetilde{X} / (\mathbb{Z}_3 \times \mathbb{Z}_3)$. Therefore, we are only interested in polynomials that are invariant under our group action. Let us start with the group action on the homogeneous coordinates $(x_0, x_1, x_2, t_0, t_1, y_0, y_1, y_2)$ of $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$. The two generators defined in eqs. (6.2a) and (6.2b) can be represented by the 8×8 matrices

$$\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} . \tag{6.10}$$

One can easily check that $[\gamma_1, \gamma_2] \neq 0$ and, in fact, the γ_1 and γ_2 actions commute up to multiplication by the central[12] matrix

$$\delta = \mathrm{diag}(\omega, \omega, \omega, 1, 1, \omega, \omega, \omega). \tag{6.11}$$

In other words, the homogeneous coordinates

$$\begin{aligned} &\mathrm{span}\,\big\{x_0, x_1, x_2, t_0, t_1, y_0, y_1, y_2\big\} \\ &\qquad\qquad = H^0\Big(\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}(1,0,0) \oplus \mathcal{O}(0,1,0) \oplus \mathcal{O}(0,0,1)\Big) \end{aligned} \tag{6.12}$$

[11] Note, however, that $\lambda_1 = \lambda_2 = \lambda_3 = 0$ is singular. A non-singular choice of complex structure is, for example, $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 1$.

[12] Commuting with γ_1 and γ_2.

of $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ carry a representation of a Heisenberg group Γ, which is the central extension

$$0 \longrightarrow \mathbb{Z}_3 \longrightarrow \Gamma \xrightarrow{\chi_1 \times \chi_2} \mathbb{Z}_3 \times \mathbb{Z}_3 \longrightarrow 0. \tag{6.13}$$

Note that the map $\chi_1 \times \chi_2$ is defined in terms of the two characters

$$\begin{aligned} \chi_1(\gamma_1) =& e^{\frac{2\pi i}{3}}, & \chi_1(\gamma_2) =& 1, & \chi_1(\delta) =& 1, \\ \chi_2(\gamma_1) =& 1, & \chi_2(\gamma_2) =& e^{\frac{2\pi i}{3}}, & \chi_2(\delta) =& 1 \end{aligned} \tag{6.14}$$

of Γ, which will be important in the following. As discussed previously for quintics, section 4.2, not all line bundles are $\mathbb{Z}_3 \times \mathbb{Z}_3$-equivariant. However, computing the polynomials invariant under the Heisenberg group Γ is sufficient for our purposes. The Γ-invariants are automatically the $\mathbb{Z}_3 \times \mathbb{Z}_3$-invariant sections of $\mathbb{Z}_3 \times \mathbb{Z}_3$-equivariant line bundles. Their number $\hat{N}^{\Gamma}_{(a_1,b,a_2)}$ in each multi-degree (a_1, b, a_2) can be read off from the multi-variable Molien series [52],

$$\begin{aligned} P\Big(\mathbb{C}[x_0,x_1,x_2,t_0,t_1,y_0,y_1,y_2]^{\Gamma}, (x,t,y)\Big) &= \sum_{a_1,b,a_2} \hat{N}^{\Gamma}_{(a_1,b,a_2)} x^{a_1} t^b y^{a_2} \\ &= \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \frac{1}{\det\Big(1 - \gamma\, \mathrm{diag}(x,x,x,t,t,y,y,y)\Big)} \\ &= 1 + t + t^2 + 2t^3 + 2x^3 + 2y^3 + 2x^2y + 2xy^2 + 2t^4 + \cdots . \end{aligned} \tag{6.15}$$

However, to construct the Hironaka decomposition it is sufficient to determine the number of invariant linearly independent polynomials of total degree a_1+b+a_2. The corresponding Poincaré series can be obtained from eq. (6.15) by setting $x = t = y = \tau$,

$$\begin{aligned} P\Big(\mathbb{C}[x_0,x_1,x_2,t_0,t_1,y_0,y_1,y_2]^{\Gamma}, \tau\Big) &= \sum_k \hat{N}^{\Gamma}_k \tau^k \\ &= \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \frac{1}{\det(1-\gamma\tau)} \\ &= 1 + \tau + \tau^2 + 10\tau^3 + 16\tau^4 + 22\tau^5 + 85\tau^6 + 142\tau^7 + 199\tau^8 + 488\tau^9 + \cdots . \end{aligned} \tag{6.16}$$

Next, we need to choose $3 + 2 + 3 = 8$ primary invariants. Similarly to the quintic case in section 4.3, we choose our primary invariants to be of the lowest possible degree. It is not hard to check that homogeneous polynomials

$$\theta_1 = t_0 \qquad \theta_2 = t_1^3 \tag{6.17a}$$

$$\theta_3 = x_0x_1x_2 \qquad \theta_4 = x_0^3 + x_1^3 + x_2^3 \tag{6.17b}$$

$$\theta_5 = y_0y_1y_2 \qquad \theta_6 = y_0^3 + y_1^3 + y_2^3 \tag{6.17c}$$

$$\theta_7 = x_0^3x_1^3 + x_0^3x_2^3 + x_1^3x_2^3 \qquad \theta_8 = y_0^3y_1^3 + y_0^3y_2^3 + y_1^3y_2^3. \tag{6.17d}$$

can be chosen as our primary invariants. They are, in fact, the choice with the lowest degrees. Rewriting eq. (6.16) as a fraction with the denominator corresponding to our choice of the primary invariants, we get

$$
\begin{aligned}
&P\Big(\mathbb{C}[x_0,x_1,x_2,t_0,t_1,y_0,y_1,y_2]^\Gamma,\tau\Big) \qquad (6.18)\\
&\quad=\frac{1}{(1-\tau)(1-\tau^3)^5(1-\tau^6)^2}\Big(1+4\tau^3+6\tau^4+6\tau^5+26\tau^6+27\tau^7+27\tau^8+46\tau^9+\\
&\qquad\qquad +42\tau^{10}+42\tau^{11}+26\tau^{12}+27\tau^{13}+27\tau^{14}+4\tau^{15}+6\tau^{16}+6\tau^{17}+\tau^{18}\Big).
\end{aligned}
$$

Thus, the number of secondary invariants is

$$
\begin{aligned}
\frac{3^5 6^2}{|\Gamma|} = 324 = 1+4\ &+6+6+26+27+27+46\\
&+42+42+26+27+27+4+6+6+1. \qquad (6.19)
\end{aligned}
$$

Notice that the polynomials in eq. (6.17) are homogeneous of multi-degree (a_1,b,a_2). Since the group action eq. (6.10) does not mix the degrees, it follows that the secondary invariants will also be homogeneous polynomials. They are, moreover, separately homogeneous in the variables $[x_0:x_1:x_2]$, $[t_0:t_1]$, and $[y_0:y_1:y_2]$. Here, we present the first few secondary invariants

$$\eta_1 = 1, \qquad (6.20\text{a})$$

$$
\begin{aligned}
\eta_2 &= x_2y_0y_1+x_1y_0y_2+x_0y_1y_2, & \eta_3 &= x_0y_0^2+x_1y_1^2+x_2y_2^2,\\
\eta_4 &= x_1x_2y_0+x_0x_2y_1+x_0x_1y_2, & \eta_5 &= x_0^2y_0+x_1^2y_1+x_2^2y_2,\\
\eta_6 &= t_1y_0y_1^2+t_1y_0^2y_2+t_1y_1y_2^2, & \eta_7 &= x_1t_1y_0y_1+x_0t_1y_0y_2+x_2t_1y_1y_2,\\
\eta_8 &= x_2t_1y_0^2+x_0t_1y_1^2+x_1t_1y_2^2, & \eta_9 &= x_0x_2t_1y_0+x_0x_1t_1y_1+x_1x_2t_1y_2, \qquad (6.20\text{b})\\
\eta_{10} &= x_1^2t_1y_0+x_2^2t_1y_1+x_0^2t_1y_2, & \eta_{11} &= x_0x_1^2t_1+x_0^2x_2t_1+x_1x_2^2t_1,\\
\eta_{12} &= t_1^2y_0^2y_1+t_1^2y_1^2y_2+t_1^2y_0y_2^2, & \eta_{13} &= x_1t_1^2y_0^2+x_2t_1^2y_1^2+x_0t_1^2y_2^2,\\
&\ \vdots & &\ \ddots
\end{aligned}
$$

We list the number of secondary invariants for a given degree in table 4. Thus we obtain the following Hironaka decomposition for the ring of Γ-invariant polynomials,

$$\mathbb{C}[x_0,x_1,x_2,t_0,t_1,y_0,y_1,y_2]^\Gamma = \bigoplus_{i=1}^{324}\eta_i\mathbb{C}[\theta_1,\dots,\theta_8]. \qquad (6.21)$$

Finally, we need to restrict the invariant ring eq. (6.21) to the complete intersection threefold $\widetilde{X}$. In other words, one must mod out the invariant ideal

$$\big\langle \widetilde{P},\widetilde{R}\big\rangle^\Gamma = \big\langle \widetilde{P},\widetilde{R}\big\rangle \cap \mathbb{C}[x_0,x_1,x_2,t_0,t_1,y_0,y_1,y_2]^\Gamma \qquad (6.22)$$

generated by the complete intersection equations $\widetilde{P}=0=\widetilde{R}$.

Since $\widetilde{P}$ is invariant, the ideal generated by $\widetilde{P}$ is just the invariant ring multiplied by $\widetilde{P}$,

$$\big\langle \widetilde{P}\big\rangle^\Gamma = \bigoplus_{i=1}^{324}\widetilde{P}\,\eta_i\,\mathbb{C}[\theta_1,\dots,\theta_8]. \qquad (6.23)$$

$\deg(\eta)$	# of η
0	1
3	4
4	6
5	6
6	26
7	27
8	27
9	46
10	42
11	42
12	26
13	27
14	27
15	4
16	6
17	6
18	1

$\deg(\eta)$	# of η
$(0,0,0)$	1
$(1,0,2)$	2
$(2,0,1)$	2
$(0,1,3)$	1
$(1,1,2)$	2
$(2,1,1)$	2
$(3,1,0)$	1
$(0,2,3)$	1
$(1,2,2)$	2
$(2,2,1)$	2
$(3,2,0)$	1
$(2,0,4)$	6
$(3,0,3)$	8
$\vdots$	$\vdots$

Table 4: Degrees of the 324 secondary invariants $\eta_1, \ldots, \eta_{324}$. On the left, we list the number of secondary invariants by total degree. On the right, we list some of invariants by their three individual (a_1, b, a_2)-degrees.

However, the ideal generated by $\widetilde{R}$ is not as simple. From eqs. (6.6) and (6.14) we see that $\widetilde{R}$ transforms like the character χ_1. Thus the elements of the invariant ring that are divisible by $\widetilde{R}$ must also be divisible by a χ_1^2-transforming polynomial (like t_1^2, for example). One can generalize the Molien formula eq. (6.16) to count these "covariant" polynomials transforming like χ_1^2 [53], namely

$$\begin{aligned} P\Big(\mathbb{C}[x_0,x_1,x_2,t_0,t_1,y_0,y_1,y_2]^{\chi_1^2},\tau\Big) &= \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\frac{\chi_1(\gamma)^2}{\det(1-\gamma\chi_1(\gamma)^2\tau)} \\ &= \tau^2+7\tau^3+13\tau^4+22\tau^5+79\tau^6+136\tau^7+199\tau^8+478\tau^9\ldots \end{aligned} \tag{6.24}$$

Choosing the same primary invariants as previously, eq.(6.17), one can rewrite eq.(6.24) as

$$\begin{aligned} &P\Big(\mathbb{C}[x_0,x_1,x_2,t_0,t_1,y_0,y_1,y_2]^{\chi_1^2},\tau\Big) \\ &\quad= \frac{1}{(1-\tau)(1-\tau^3)^5(1-\tau^6)^2}\Big(\tau^2+6\tau^3+6\tau^4+4\tau^5+27\tau^6+27\tau^7+26\tau^8+42\tau^9 \\ &\qquad +42\tau^{10}+46\tau^{11}+27\tau^{12}+27\tau^{13}+26\tau^{14}+6\tau^{15}+6\tau^{16}+4\tau^{17}+\tau^{20}\Big). \end{aligned} \tag{6.25}$$

Summing the coefficients in the numerator, we see that we again get the same number $(= 324)$ of secondary χ_1^2-covariant generators. This is expected since we are using the same primary invariants. The first few secondary χ_1^2-covariants are:

$$\eta_1^{\chi_1^2} = t_1^2, \tag{6.26a}$$

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$$
\begin{aligned}
\eta_2^{\chi_1^2} &= x_0^2 x_2 + x_0 x_1^2 + x1 x_2^2, & \eta_3^{\chi_1^2} &= y_1 y_2^2 + y_0 y_1^2 + y_0^2 y_2,\\
\eta_4^{\chi_1^2} &= x_2 y_1 y_2 + x_0 y_0 y_2 + x_1 y_0 y_1, & \eta_5^{\chi_1^2} &= x_2 y_0^2 + x_0 y_1^2 + x_1 y_2^2,\\
\eta_6^{\chi_1^2} &= x_0^2 y_2 + x_2^2 y_1 + x_1^2 y_0, & \eta_7^{\chi_1^2} &= x_0 x_1 y_1 + x_0 x_2 y_0 + x_1 x_2 y_2,\\
\eta_8^{\chi_1^2} &= y_0^2 y_1 + y_1^2 y_2 + y_0 y_2^2 t_1, & \eta_9^{\chi_1^2} &= x_0 x_1 t_1 y_0 + x_0 x_2 t_1 y_2 + x_1 x_2 t_1 y_1,\\
\eta_{10}^{\chi_1^2} &= x_1 t_1 y_0^2 + x_2 t_1 y_1^2 + x_0 t_1 y_2^2, & \eta_{11}^{\chi_1^2} &= x_2 t_1 y_0 y_2 + x_0 t_1 y_0 y_1 + x_1 t_1 y_1 y_2,\\
\eta_{12}^{\chi_1^2} &= x_0 x_2^2 t_1 + x_0^2 x_1 t_1 + x_1^2 x_2 t_1, & \eta_{13}^{\chi_1^2} &= x_1^2 t_1 y_2 + x_2^2 t_1 y_0 + x_0^2 t_1 y_1,\\
\eta_{14}^{\chi_1^2} &= x_2 t_1^2 y_2^2 + x_0 t_1^2 y_0^2 + x_1 t_1^2 y_1^2, & \eta_{15}^{\chi_1^2} &= x_2 t_1^2 y_0 y_1 + x_0 t_1^2 y_1 y_2 + x_1 t_1^2 y_0 y_2,\\
&\vdots & &\ddots
\end{aligned}
\tag{6.26b}
$$

Hence, the space of χ_1^2-covariant polynomials, that is, transforming like χ_1^2, is given by the "equivariant Hironaka decomposition" [53] (compare with eq. (6.21))

$$
\mathbb{C}[x_0, x_1, x_2, t_0, t_1, y_0, y_1, y_2]^{\chi_1^2} = \bigoplus_{i=1}^{324} \eta_i^{\chi_1^2}\, \mathbb{C}[\theta_1, \dots, \theta_8]. \tag{6.27}
$$

To summarize, even though $\widetilde{R}$ is not invariant, it generates an ideal which contains Γ-invariant polynomials. Using the above generalization of the Hironaka decomposition, a basis for these invariants is

$$
\big\langle \widetilde{R} \big\rangle^{\Gamma} = \bigoplus_{i=1}^{324} \widetilde{R}\, \eta_i^{\chi_1^2}\, \mathbb{C}\left[\theta_1, \dots, \theta_8\right]. \tag{6.28}
$$

6.3 Quotient ring

By the results of the previous section, we know for any fixed multi-degree (a_1, b, a_2):

- A (finite) basis for the Γ-invariant polynomials

$$
I = \mathbb{C}\big[\vec{x}, \vec{t}, \vec{y}\big]^{\Gamma}_{(a_1,b,a_2)}. \tag{6.29}
$$

  In particular, the polynomials are linearly independent of each other.

- Generators for the Γ-invariant ideal generated by the complete intersection eqs. (5.3a) and (5.3b),

$$
\begin{aligned}
J &= \big\langle \widetilde{P}, \widetilde{R} \big\rangle^{\Gamma}_{(a_1,b,a_2)} = \big\langle \widetilde{P} \big\rangle^{\Gamma}_{(a_1,b,a_2)} + \big\langle \widetilde{R} \big\rangle^{\Gamma}_{(a_1,b,a_2)}\\
&= \Big\langle \widetilde{P} \cdot \mathbb{C}\big[\vec{x}, \vec{t}, \vec{y}\big]^{\Gamma}_{(a_1-3,b-1,a_2)},\ \widetilde{R} \cdot \mathbb{C}\big[\vec{x}, \vec{t}, \vec{y}\big]^{\chi_2^2}_{(a_1,b-1,a_2-3)} \Big\rangle_{(a_1,b,a_2)}.
\end{aligned}
\tag{6.30}
$$

  The generating polynomials of J are not automatically linearly independent.

It remains to find a basis for the quotient

$$
\Big(\mathbb{C}\big[\vec{x}, \vec{t}, \vec{y}\big] \Big/ \big\langle \widetilde{P}, \widetilde{R} \big\rangle \Big)^{\Gamma}_{(a_1,b,a_2)} = \mathbb{C}\big[\vec{x}, \vec{t}, \vec{y}\big]^{\Gamma}_{(a_1,b,a_2)} \Big/ \big\langle \widetilde{P}, \widetilde{R} \big\rangle^{\Gamma}_{(a_1,b,a_2)} = I/J, \tag{6.31}
$$

(a_1, b, a_2)	$\hat{N}^\Gamma$	N^Γ	(a_1, b, a_2)	$\hat{N}^\Gamma$	N^Γ	(a_1, b, a_2)	$\hat{N}^\Gamma$	N^Γ
(2,1,1)	4	4	(2,27,1)	56	56	(5,6,1)	49	37
(2,2,1)	6	6	(2,28,1)	58	58	(5,7,1)	56	42
(2,3,1)	8	8	(2,29,1)	60	60	(5,8,1)	63	47
(2,4,1)	10	10	(2,30,1)	62	62	(5,9,1)	70	52
(2,5,1)	12	12	(2,31,1)	64	64	(5,10,1)	77	57
(2,6,1)	14	14	(2,32,1)	66	66	(5,11,1)	84	62
(2,7,1)	16	16	(2,33,1)	68	68	(5,12,1)	91	67
(2,8,1)	18	18	(2,34,1)	70	70	(5,1,4)	70	53
(2,9,1)	20	20	(3,1,3)	23	20	(6,1,3)	63	48
(2,10,1)	22	22	(3,2,3)	34	29	(6,2,3)	94	66
(2,11,1)	24	24	(3,3,3)	46	38	(7,1,2)	48	38
(2,12,1)	26	26	(3,4,3)	57	47	(7,2,2)	72	52
(2,13,1)	28	28	(3,5,3)	68	56	(7,3,2)	96	66
(2,14,1)	30	30	(3,6,3)	80	65	(8,1,1)	30	23
(2,15,1)	32	32	(4,1,2)	20	18	(8,2,1)	45	31
(2,16,1)	34	34	(4,2,2)	30	26	(8,3,1)	60	39
(2,17,1)	36	36	(4,3,2)	40	34	(8,4,1)	75	47
(2,18,1)	38	38	(4,4,2)	50	42	(8,5,1)	90	55
(2,19,1)	40	40	(4,5,2)	60	50	(8,6,1)	105	63
(2,20,1)	42	42	(4,6,2)	70	58	(10,1,2)	88	64
(2,21,1)	44	44	(4,7,2)	80	66	(11,1,1)	52	37
(2,22,1)	46	46	(5,1,1)	14	12	(11,2,1)	78	48
(2,23,1)	48	48	(5,2,1)	21	17	(11,3,1)	104	59
(2,24,1)	50	50	(5,3,1)	28	22	(11,4,1)	130	70
(2,25,1)	52	52	(5,4,1)	35	27	(14,1,1)	80	54
(2,26,1)	54	54	(5,5,1)	42	32	(14,2,1)	120	68

Table 5: All homogeneous degrees leading to few (≤ 70) invariant sections $N^\Gamma = N^\Gamma_{(a_1,b,a_2)}$ on $\widetilde{X}$. For comparison, we also list the number $\hat{N}^\Gamma = \hat{N}^\Gamma_{(a_1,b,a_2)} = \dim \mathbb{C}[\vec{x}, \vec{t}, \vec{y}]^\Gamma_{(a_1,b,a_2)}$ of invariant polynomials before quotienting out the relations generated by the complete intersection equations $\widetilde{P} = 0 = \widetilde{R}$.

corresponding to the restriction of the invariant sections on $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$ to the complete intersection $\widetilde{X}$. This is technically more difficult than the previous quotients, where we were able to use Gröbner bases or pick suitable primary invariants to find the quotient. Here, we will resort to a numerical computation of the quotient. To do this, note that the ideal elements J are linear combinations of invariants I. Hence, thinking of I, J as column vectors, there is a matrix

$$M \in \mathrm{Mat}_{|J|\times|I|}(\mathbb{C}) : \qquad MI = J. \tag{6.32}$$

The kernel of M is a basis for the quotient I/J. Of course, due to floating-point precision limits, there are generally no exact null-vectors. However, the singular value decomposi-

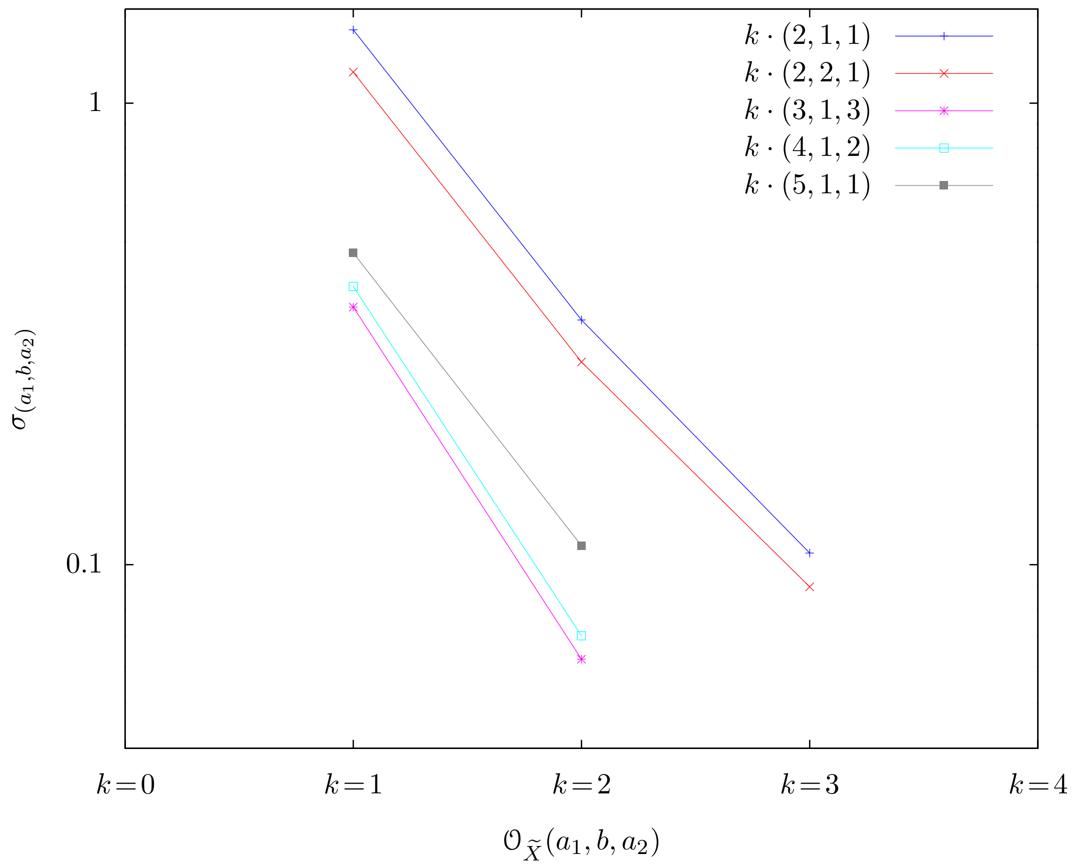

Figure 8: The error measure $\sigma_{(a_1,b,a_2)}(X)$ for the metric on the $\mathbb{Z}_3 \times \mathbb{Z}_3$-quotient X, computed for different Kähler moduli but common complex structure $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1$. Note that we chose $k = \gcd(a_1, b, a_2)$ as the independent variable, and stopped increasing k as soon as N^Γ exceeded 200. In each case we iterated the T-operator 5 times, numerically integrating using $N_p = 50{,}000$ points. Then we evaluated $\sigma_{(a_1,b,a_2)}(X)$ using 5,000 different test points.

tion [54] is a well-behaved numerical algorithm to compute an orthonormal basis for the kernel. In table 5 we list the dimension

$$N^\Gamma_{(a_1,b,a_2)} = \dim_{\mathbb{C}} \big(I/J\big) \qquad (6.33)$$

of the quotient space for various multi-degrees (a_1, b, a_2).

6.4 Results

We implemented Donaldson's algorithm to compute the Calabi-Yau metric on the threefold $X = \widetilde{X}\big/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. As discussed earlier, the convergence of the balanced metrics is essentially independent of the complex structure. Hence, we will consider an explicit example where $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1$. In figure 8 we demonstrate that the numerical metric indeed approximates the Calabi-Yau metric, as it should.

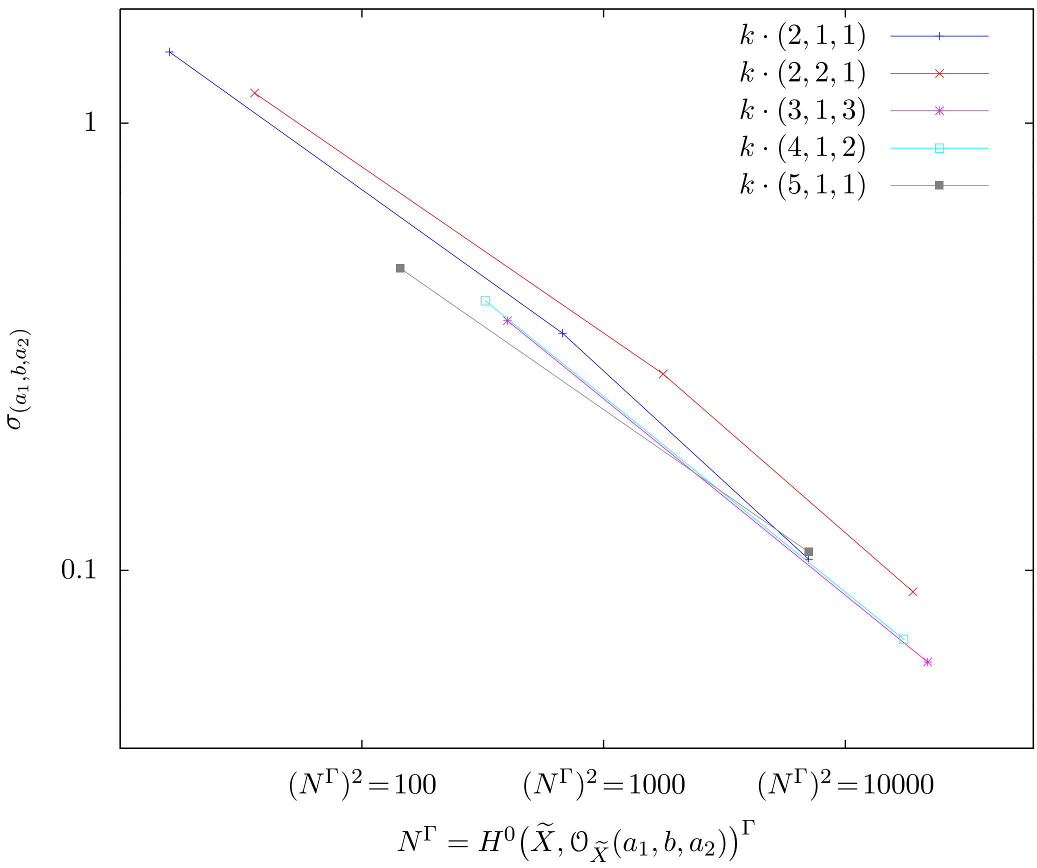

Figure 9: The same data as in figure 8, but plotted as a function of the number of free parameters $(N^\Gamma_{(a_1,b,a_2)})^2$ in the ansatz for the Kähler potential.

In contrast to the quintic, where the single Kähler modulus is the overall volume, the Schoen quotient threefold X has a $h^{1,1}(X) = 3$-dimensional Kähler moduli space, see eq. (6.9). The Kähler moduli are determined through the three independent degrees (a_1, b, a_2). Note that the integer $k = \gcd(a_1, b, a_2)$ in figure 8 serves only to measure the refinements along a ray in the Kähler moduli space. In order to properly compare the metric convergence for different rays in the Kähler moduli space, we should consider $(N^\Gamma_{(a_1,b,a_2)})^2$, which is the number of free parameters in the ansatz for the Kähler potential and, hence, measures the numerical complexity of the whole algorithm. We do this in figure 9, and see that the accuracy is essentially determined by $(N^\Gamma_{(a_1,b,a_2)})^2$, and depends only slightly on the details of the Kähler moduli.

Finally, we note again that $\sigma_{(a_1,b,a_2)}(X)$ is also the error measure for the metric pulled back to the covering space $\widetilde{X}$ of X. It is useful to compare this result with the convergence of the Calabi-Yau metric on $\widetilde{X}$ obtained directly as discussed in section 5. We have numerically performed this comparison and obtained results similar to those found in the quintic case, see figure 4. That is, when measured by the numerical effort involved, the

$\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetric method of this section is far more efficient.

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A. Primary invariants

In this appendix, we check that the invariants in eq. (4.15) can be chosen to be the primary invariants, that is, form a "homogeneous system of parameters". In fact, the following criteria are equivalent, see [39] Proposition 2.3:

- $\{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5\}$ are a homogeneous system of parameters (h.s.o.p.).
- $\dim \Big(\mathbb{C}[z_0, z_1, z_2, z_3, z_4] / \langle \theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \rangle \Big) = 0$
- The only common solution to $\theta_i = 0$, $i = 1, \ldots, 5$ is $z_0 = z_1 = z_2 = z_3 = z_4 = 0$.

Using Singular [41], we can test the dimension criterion easily:

```
                       SINGULAR                          /
 A Computer Algebra System for Polynomial Computations   /   version 3-0-1
                                                       0<
     by: G.-M. Greuel, G. Pfister, H. Schoenemann        \   October 2005
FB Mathematik der Universitaet, D-67653 Kaiserslautern    \
> ring r=0,(z0,z1,z2,z3,z4),dp;
> poly t1=z0*z1*z2*z3*z4;
> poly t2=z0^3*z1*z4+z0*z1^3*z2+z0*z3*z4^3+z1*z2^3*z3+z2*z3^3*z4;
> poly t3=z0^5+z1^5+z2^5+z3^5+z4^5;
> poly t4=z0^10+z1^10+z2^10+z3^10+z4^10;
> poly t5=z0^8*z2*z3+z0*z1*z3^8+z0*z2^8*z4+z1^8*z3*z4+z1*z2*z4^8;
> ideal i=t1,t2,t3,t4,t5;
> dim(std(i));
0
```

Hence, eq. (4.15) is indeed a h.s.o.p.

B. Implementation details

B.1 Tensors

We use the blitz++ [55] library for all tensor computations. For example, here is the ordinary (serial) computation of the T-operator:

```
template<typename Mfd>
Array<COMPLEX,2> Metric<Mfd>::Toperator_integrand
(const Point &p) const
{
  using namespace blitz;

  Array<COMPLEX,1> s_val(N), sbar_val(N);
  s_val = X.SectionsAt(p);
  sbar_val = conj(s_val);

  COMPLEX D = s_h_sbar(s_val, sbar_val); // = ||s||_h^2

  Array<COMPLEX,2> result(N, N);
  firstIndex a;  secondIndex b;
  result = s_val(a)*sbar_val(b)/D;

  return(result);
}

template<typename Mfd>
Metric<Mfd> Metric<Mfd>::Toperator_slow() const
{
  Array<COMPLEX,2> tmp(N,N); tmp = 0;
  for (typename Manifold::const_iterator p=X.begin(); p!=X.end(); p++)
    tmp += Toperator_integrand(*p) * p->weight;
  tmp *= N / X.Volume();

  Metric<Manifold> result(X);  result.h_ab = tmp;
  result.compute_hinv(); // compute h^ab
  return(result);
}
```

B.2 MPI

In order to speed up computations we use a cluster of ordinary PCs, consisting of 11 machines connected via gigabit Ethernet. Each node has one 2.2 GHz dual-core Opteron processor and 2 GiB of RAM.

The main task in computing the metrics is to compute the T-operator, see eq. (2.27). Performing the numerical integration is embarrassingly parallel, and does not even need any high-speed network connection. For the stochastic integration one has to

- Compute the (weighted) integrand of the T-operator at each point, and
- Sum the resulting matrix.

We solve this by a bag-of-tasks, where each node adds up the contribution of a few points and then asks for another work set. In the end, the partial sum computed at each node is reduced to the master node. To effectively write distributed programs on the cluster, we make use of the MPI standard implemented by OpenMPI [56]. For example, here is the parallel implementation of the T-operator:

```
template<typename Mfd>
void Metric<Mfd>::Toperator_slave(int)
{
  WorkQueue<typename Mfd::const_iterator> work(X.begin(), X.end());
  Array<COMPLEX,2> tmp(N,N); tmp = 0;
  while (work.ReceiveMoreWork()) {
    for (typename Mfd::const_iterator
     p=work.begin(); p!=work.end(); p++)
      tmp += Toperator_integrand(*p) * p->weight;
  }
  Cluster::SendSummandArray(tmp);
}

template<typename Mfd>
Metric<Mfd> Metric<Mfd>::Toperator() const
{
  ClusterExecMethod< Metric<Mfd>, &Metric<Mfd>::Toperator_slave >
    ().Run(*this);

  WorkQueue<typename Mfd::const_iterator> work(X.begin(), X.end(),20);
  work.Finish(); // main loop

  Array<COMPLEX,2> tmp(N,N);  tmp = 0;
  Cluster::ReceiveSumArray(tmp);
  tmp *= N / X.Volume();

  Metric<Manifold> result(X);
  result.h_ab = tmp;
  result.compute_hinv();
  return result;
}
```

B.3 Multivariate polynomials

Every section is, at the end of the day, some multivariate polynomial. For that reason, we implemented a C++ library for sparse multivariate polynomials. In addition to the usual arithmetic operations, it supports differentiation and can copy polynomials to remote nodes via MPI. Using this library, we can easily work with arbitrary polynomials. For example, the program to compute the metric on the Fermat quintic can, without change, also work with generic quintics that are a non-trivial sum over all 126 monomials, see figure 4.

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